NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATEIAL

MATHEMATICS

POST GRADUATE

PG (MT) : X A(1)

(PURE MATHEMATICS)

Advanced Differential Geometry

PREFACE

In the auricular structure introduced by this University for students of Post- Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so mat they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

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NETAJI SUBHAS
OPEN UNIVERSITY

PG (MT)–XA (I)

Advanced Differential Geometry

UNIT - 1

§ 1.1 Calculus on R^n :

Let R denote the set of real numbers. For an integer n > 0, let R^n be the cartesian product

$$\underbrace{R \times R \times R \times \cdots \times R}_{\text{n times}}$$

of the set of all ordered n-tuples $(x^1,\cdots,x^n)$ of real numbers. Individual n-tuple will be denoted at times by a single letter, e.g. $x=(x^1,\cdots,x^n), y=(y^1,\cdots,y^n)$ and so on.

Co-ordinate functions : Let $x=(x^1,x^2,{}_i\, x^n)\in R^n$. Then, the functions u_i : $R^n \to R$ defined by $u_i(x^1,x^2,\cdots x^i\cdots x^n)=x^i$

We are now going to define a function to be differentiable of class C^∞.

A real-valued function $f: U \subset R^n \to R$,

U being an open set of R^n, is said to be of class c^k if

i) all its partial derivatives of order less than or equal to k exist and

ii) are continuous functions at every point of U.

By class C^0, we mean that f is merely continuous from U to R. By class C^∞, we mean that that partial derivatives of all orders of f exist and are continuous at every point of U. In this case, f is said to be a smooth function.

Note : By class C^ω on U, we mean that f is real analytic on U i.e. expandable in a power series about each point on U. A C^ω function is a C^∞ function but the converse is not true.

Exercise : 1. Let $f: R \to R$ be defined by

$$f(x)=e^{-\frac{1}{x^2}}, \qquad x \neq 0$$
$$=0, \qquad x=0$$

Show that f is a differentiable function of class C^∞.

Solution : Note that

$$f'(o)=\lim_{h\to 0}\frac{f(o+h)-f(o)}{h}=\lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}}{h}$$

Apply L'Hospital's Rule, on taking, $h=\frac{1}{u}$ we see that $h\to o$ gives $u\to\infty$

$$\therefore\ f'(o) = \lim_{u \to \infty} u.e^{-u^2}$$

$$= \lim_{u \to \infty} \frac{u}{e^{u^2}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{u \to \infty} \frac{1}{2ue^{u^2}}$$

$$= \lim_{u \to \infty} \frac{e^{-u^2}}{2u}$$

$$= 0$$

Again, $f'(x) = 2x^{-3} e^{-\frac{1}{x^2}}, \quad x \neq 0$

$\therefore\ f''(o) = \lim_{h \to 0} \frac{f'(0+h) - f'(0)}{h}$ and on putting $\frac{1}{h} = u$, we get

$$f''(o) = \lim_{u \to \infty} \frac{2u^4}{e^{u^2}} \left(\frac{\infty}{\infty}\right)$$

Applying L' Hospital rule successively, we find

$$f''(0) = \lim_{u \to \infty} \frac{8u^3}{2ue^{u^2}}$$

$$= \lim_{u \to \infty} \frac{4u^2}{e^{u^2}}$$

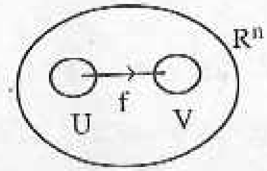
$$= \lim_{u \to \infty} \frac{8u}{2ue^{u^2}}$$

$$= \lim_{u \to \infty} \frac{4}{e^{u^2}}$$

$$= 0$$

Proceding in this manner, we can show that.

$f^n(0) = 0$, for $n = 1, 2, \cdots$



Hence f is a function of class C^∞.

A mapping $f : U \to V$

of an open set $U \subset R^n$ to an open set $V \subset R^n$ is called a homeomorphism if

i) f is bijective i.e. one to one and onto, as well as

ii) f, f^{-1} are continuous.

Exercise : 2. Let $f : R \to R$ be such that

$$f(x) = 5x + 3$$

Show that f is a homoeomorphism on R.

3. Let $f : R \to R$ be defined by

$$f(x) = x^3$$

Test i) whether f is a differentiable function of class C^∞ or not

ii) whether f is a homeomorphism or not.

[**Ans.** : i) f is of class C^∞. ii) f is homeomorphism]

Solution : 2. Note that

$f(x) - f(y) = 5(x - y)$

$\therefore f(x) = f(y)$ if and only if $x = y$

Hence f is one one.

Let $y = 5x + 3$

$\therefore \quad x = \dfrac{y-3}{5}$

and hence $f^{-1}: R \to R$ is defined as

$$f^{-1}(y) = \frac{y-3}{5}$$

Again, $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$, Thus f is onto.

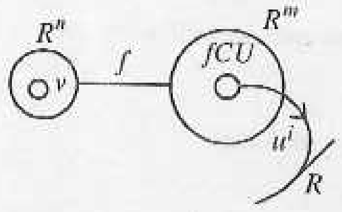
Consequently f is bijective.

Both f, f^{-1} are continuous functions, (being polynomial functions) f is a homeomorphism on R.

Note : (i) If $f : U \subset R^n \to R^m$ is a mapping, such that

$$f(x^1, \cdots x^n) = \left(f^1(x^1, \cdots, x^n), \cdots, f^m(x^1, \cdots, x^n)\right)$$

where $f^j(x) = u^j \circ f,\ 1 \le j \le m,\ u^j$ being co-ordinate functions on R^m



we define the <u>Jacobian matrix</u> of f at $(x^1, \cdots, x^n)$, denoted by J, as

$$J = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}.$$

(ii) In particular, when $m = n$ i.e., if $f : U \subset R^n \to R^n$ is a mapping such that,

if $f = (f^1, f^2, \cdots, f^n)$ has continuous partial derivatives i.e. if each f^i $i = 1, 2, \cdots, n$, has continuous partial derivatives on U, we say that f is continuously differentiable on $U \subset R^n$.

(iii) If $f = (f^1, \cdots, f^n)$ is continuously differentiable on $U \subset R^n$ and the Jacobian is non-zero, then f is one-one on U.

Exercise : 4. Consider the mapping

$$\phi : R^2 \to R^2$$

given by

$$\phi : \quad y^1 = x^1 \cos x^2$$
$$y^2 = x^1 \sin x^2$$

Show that ϕ is one-to-one on a sufficiently small neighbourhood of each point (x^1, x^2) of R^2 with $x^1 \neq 0$.

Solution : The given mapping

$$\phi = (\phi^1, \phi^2): R^2 \to R^2 \text{ is given by } \phi^1 = x^1 \cos x^2, \ \phi^2 = x^1 \sin x^2$$

Then, we have

$$\frac{\partial\phi^1}{\partial x^1} = \cos x^2, \ \frac{\partial\phi^1}{\partial x^2} = -x^1 \sin x^2, \ \frac{\partial\phi^2}{\partial x^1} = \sin x^2, \ \frac{\partial\phi^2}{\partial x^2} = x^1 \cos x^2$$

Hence each $\frac{\partial\phi^i}{\partial x^j}$, $i, j = 1,2$ is continuous for all values of x^1 and x^2 in R^2. Thus ϕ is continuously differentiable on R^2.

Again the Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial\phi^1}{\partial x^1} & \frac{\partial\phi^1}{\partial x^2} \\ \frac{\partial\phi^2}{\partial x^1} & \frac{\partial\phi^2}{\partial x^2} \end{vmatrix} = x^1 \neq 0 \text{ if and only if } x^1 \neq 0 \text{ in } R^2.$$

Consequently, ϕ is one-to-one on a sufficiently small neighbourhood of each point (x^1, x^2) of R^2 with $x^1 \neq 0$.

A mapping

$$f : U \to V$$

of an open set $U \subset R^n$ onto an open set $V \subset R^n$ is called a <u>C^k – diffeomorphism</u>, $k \geq 1$ if

i) f is a homeomorphism of U onto V and

ii) f, f^{-1} are of class C^k.

when f is a C^∞ – diffeomorphism, we simply say <u>diffeomorphism</u>.

Exercise : 5. Let $\phi : R^2 \to R^2$ be defined by

$$\phi(u, v) = (ve^u, u)$$

Determine whether ϕ is a diffeomorphism or not.

6. Let $\phi : R^2 \to R^2$ be defined by

$$\phi(x^1, x^2) = (x^1 e^{x^2} + x^2,\ x^1 e^{x^2} - x^2)$$

Show that ϕ is a diffeomorphism. [**Ans.** : 5. ϕ is a diffeomorphism]

For $i = 1, \cdots, n$; let $u^i : R^n \to R$

be the coordinate functions on R^n i.e. for every $p \in R^n$

1. 1) $u^i(p) = p^i$ where $p = (p^1, \cdots, p^n)$

Such $u^i s$ are continuous functions from $R^n \to R$.. We call this n-tuple of functions $(u^1, u^2, \cdots, u^n)$ the <u>standard co-ordinate system</u> of R^n.

If $f : U \subset R^n \to R^n$

is a mapping defined on $U \subset R^n$, then, f is determined by its co-ordinate functions $(f^1, \cdots, f^n)$ where

1.2) $f^i = u^i \circ f$, $i = 1, \cdots, n$

and each $f^i : U \subset R^n \to R$ are real valued functions, defined on an open subset U of R^n.

Thus for every $p \in U \subset R^n$

$$f^i(p) = (u^i \circ f)(p) = u^i(f(p)) \text{ where } f(p) = q = (q^1, \cdots, q^n)$$

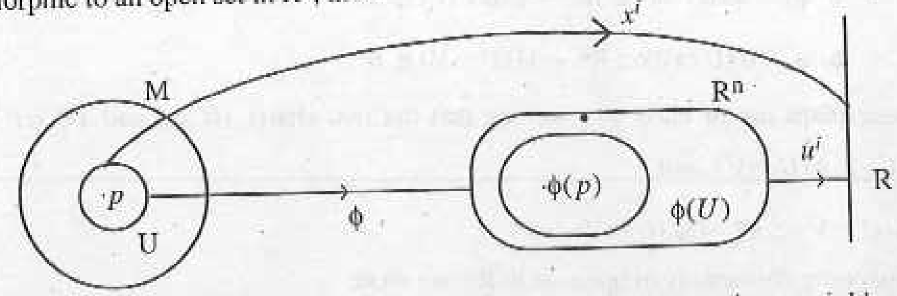
$$= u^i(q^i, \cdots, q^n) = q^i \text{ by 1.1)}$$

1.3) consequently $f(p) = \left(f^1(p), f^2(p), \cdots, f^n(p)\right)$, $\forall\, p \in U \subset R^n$

The map f is of class c^k if each of its co-ordinate functions $f^i : i = 1, \cdots, n$ is of class c^k.

§ 1.2 Differentiable Mainfold :

Let M be a Hausdorff, second countable space. If every point of M has a neighbourhood homeomorphic to an open set in R^n, then



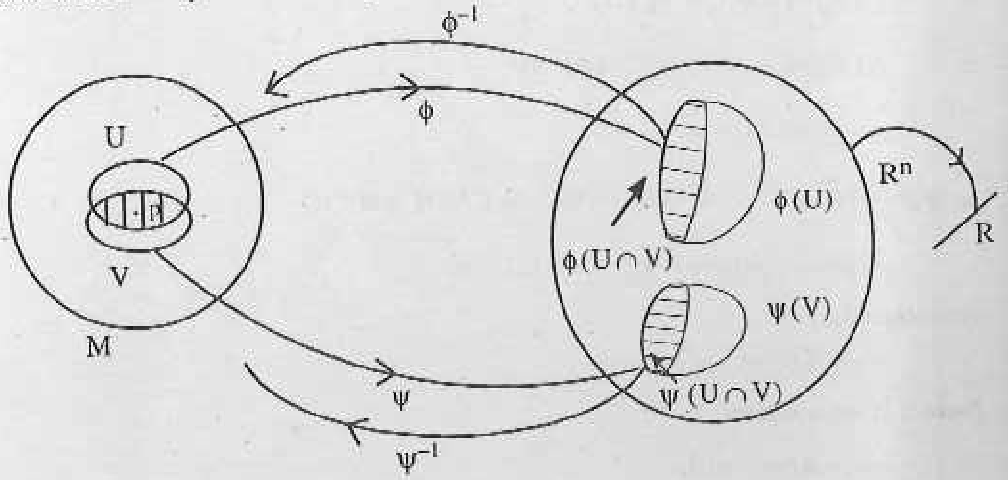
M is said to be a manifold. Thus in a manifold for each $p \in M$, there exists a neighbourhood U of $p \in M$ and a homeomorphism ϕ of U onto an open subset of R^n. The pair (U, ϕ) is called a chart.

Each such chart (U, ϕ) on M induces a set of n real valued functions on U defined by

2.1) $x^i = u^i \circ \phi$, $i = 1, 2, \cdots n$

where u^i, s are defined by (1.1) and it is to be noted that whatever be the point p and the neighbourhood $U, u^i, i = 1, 2, \cdots n$ always represent co-ordinate functions. The functions $(x^1, x^2, \cdots, x^n)$ are called coordinate functions or a coordinate system on U and U is called the domain of the coordinate system. The chart (U, ϕ) is sometimes called an n-coordinate chart.

Let (V, ψ) be another chart of p, which overlaps the previous chart (U, ϕ). Let $(y^1, \cdots, y^n)$ be local coordinate system on V of p, so that



2.2) $y^i = u^i \circ \psi, \quad, i = 1, 2, \cdots, n$

We can construct two composite maps

2.3) $\phi \cdot \psi^{-1} : \psi(U \cap V) \subset R^n \to \phi(U \cap V) \subset R^n$

$$\phi \cdot \psi^{-1} : \phi(U \cap V) \subset R^n \to \psi(U \cap V) \subset R^n$$

If these maps are of class c^k, we say that the two charts (U, ϕ) and (V, ψ) are c^k-related. If $q \in \phi(U \cap V)$ and

$$g : \phi(U \cap V) \subset R^n \to \psi(U \cap V) \subset R^n$$

is a mapping defined on an open set in R^n, we write

2.4) $g(q) = \psi(\phi^{-1}(q))$.

Exercise : 1 Find a functional relation between the two local coordinate systems defined in the overlap region of any point of a manifold M.

Solution : given that

$$q \in \phi(U \cap V),$$

$$g(q) = (\psi \circ \phi^{-1})(q) \text{ by 2.4)}$$

Let $\phi(p) = q$, where $p \in U \cap V$. Then

$$g(\phi(p)) = (\psi \circ \phi^{-1})(\phi(p)) = \psi(p)$$

or $\quad u^i(g(\phi(p))) = u^i(\psi(p)), \; i = 1, 2, \cdots, n$

or $\quad g^i(\phi(p)) = \psi^i(p)$ by 1.1)

or $\quad g^i(x^1(p), \cdots, x^n(p)) = y^i(p)$, as

$$x^i(p) = u^i(\phi(p)) = \phi^i(p)$$

$\therefore \; \phi(p) = (\phi^1(p)), \cdots, \phi^n(p) = (x^1(p), \cdots, x^n(p))$ and

$$y^i(p) = u^i(\psi(p)) = \psi^i(p) \; i = 1, 2, \cdots n.$$

consequently,

$$y^i = q^i(x^1, x^2, \cdots, x^n)$$

Note : If we consider

$$g(q) = \phi(\psi^{-1}(q)),$$

then one finds $x^i = g^i(y^1, y^2, \cdots, y^n)$, $i = 1, \cdots, n$

A collection $\Omega = \{(U_i, \phi_i)\}, i \in A$, (an index set) of c^k related charts are said to be maximal collection if a co-ordinate pair (V, ψ), c^k related with every chart is also a member of Ω.

A maximal collection of c^k-related charts is called a c^k-atlas. A c^k n-dimensional differentiable manifold M is an n-dimensional manifold M together with a c^k-atlas.

Unless otherwise stated, we shall consider a differentiable manifold of class C^∞.

Examples : 1. R^n with the usual topology is an example of a differentiable manifold with respect to the atlas (U, ϕ) where U = R^n and ϕ = the identity transformation.

2. Let S^1 be the circle in the *xy* plane R^2, centered at the origin and of radius 1. We give S^1, the topology of a subspace of R^2. Let

$$U_1 = \{p = (x, y) \in s^1 | y > 0\}$$

$$U_2 = \{p = (x, y) \in s^1 | y < 0\}$$

$$U_3 = \{p = (x, y) \in s^1 | x > 0\}$$

$$U_4 = \{p = (x, y) \in s^1 | x < 0\}$$

Then each U_i is an open subset of S^1 and $S' = \underset{i}{\cup} U_i, i = 1, 2, 3, 4$

Let I = (–1, 1) be an open interval of R and we define

$\phi_1 : U_1 \to I \subset R$ be such that

$\phi_1(x, y) = x$ i.e. $\phi_1^{-1}(x) = (x, y), y > 0$

$\phi_2 : U_2 \to I \subset R$ be such that

$\phi_2(x, y) = x$ i.e. $\phi_2^{-1}(x) = (x, y), y < 0$

$\phi_3 : U_3 \to I \subset R$ be such that

$\phi_3(x, y) = y$ i.e. $\phi_3^{-1}(y) = (x, y), x > 0$

$\phi_4 : U_4 \to I \subset R$ be such that

$\phi_4(x, y) = y$ i.e. $\phi_4^{-1}(y) = (x, y), x < 0$

Note that each ϕ_i is a homeomorphism on R and thus each (u_i, ϕ_i) is a chart of S'. Now $U_1 \cap U_2 = \phi$, $U_1 \cap U_3 = 1^{st}$ quadrant, $U_1 \cap U_4 = 2^{nd}$ quadrant, $U_2 \cap U_3 = 4^{th}$ quadrant, $U_2 \cap U_4 = 3^{rd}$ quadrant.

Then

$A = \{(U_i, \phi_i) : i = 1, 2, 3, 4)$ is an atlas of s^1

As $U \cap U_3 \neq \phi$, let $p \in U_1 \cap U_3$, then

$(\phi_1 \circ \phi_3^{-1})(y) = \phi_1(x, y) = x$ and

$(\phi_3 \circ \phi_1^{-1})(x) = \phi_3(x, y) = y$

Thus each $\phi_1 \circ \phi_3^{-1}$ and $\phi_3 \circ \phi_1^{-1}$ is of class C^∞. Similarly, it can be shown that each $\phi_1 \circ \phi_4^{-1}$, $\phi_4 \circ \phi_1^{-1}$, $\phi_2 \circ \phi_3^{-1}$, $\phi_3 \circ \phi_2^{-1}$, $\phi_2 \circ \phi_4^{-1}$, $\phi_4 \circ \phi_2^{-1}$, is of class C^∞ and hence s^1 is an one dimensional differentiable manifold with an atlas $\{(U_i, \phi_i)\}_{i=1,2,3,4}$

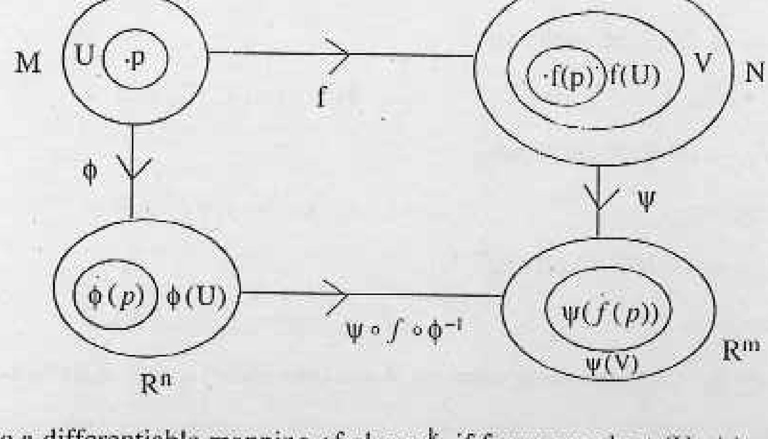
Exercise : 2. Let (M^n, A) be a differentiable manifold with a C^∞ atlas A. Let $p \in M$. Then there exists $(U, \phi) \in A$ such that $p \in U$ and $\phi(p) = 0$.

Note : 1. It is to be noted that every second countable, Hausdorff Space M admits partitions of unity. Partitions of unity admits Riemannian metric. Our aim is to study a Riemannian Manifold and for this reason we consider such topological spaces for a manifold.

2. It is enough to consider only a topological space for studying mainfold.

§ 1.3. Differentiable Mapping :

Let M be an n-dimensional and M be an m-dimensional differentiable manifold. A mapping $f : M \to N$.



is said to be a differentiable mapping of class c^k, if for every chart (U, ϕ) containing p of M and every chart (V, ψ) containing f(p) of N

3.1) i) $f(U) \subset V$ and

ii) the mapping $\psi \circ f \circ \phi^{-1} : \phi(U) \subset R^n \to \psi(V) \subset R^m$ is of class c^k.

By a differentiable mapping, we shall mean, unless otherwise stated, a mapping of class C^∞.

If $(x^1, \cdots, x^n)$ and $(y^1, \cdots, y^m)$ are respectively the local coordinate systems defined in a neighbourhood U of p of M and V of f(p) of N, then it can be shown, as done earlier

3.2) $y^j \circ f = g^j(x^1, \cdots, x^n), \ j = 1, \cdots, m$

where g is a differentiable function defined on $V \subset N$ and

3.3) $g(q) = (\psi \circ f \circ \phi^{-1})(q), \ q \in \phi(U)$.

Let M and N be two n-dimensional differentiable manifolds. A mapping

$$f : M \to N$$

is called a diffeomorphism if

i) f and f^{-1} are differentiable mappings of class C^∞

ii) f is a bijection

In such cases, M and N are said to be diffeomorphic to each other.

Exercise : 1. Let M and N be two differentiable manifolds with M=N=R. Let (U, ϕ) and (V, ψ) be two charts on M and N respectively, where

$U = R$

$\phi : U \to R$ be the identity mapping and

$V = R$

$\psi : V \to R$ be the mapping defined by

$\psi(x) = x^3$.

Show that the two structures defined on R are not C^∞-related even though M and N are diffeomorphic where

$$f : M \to N$$

is defined by

$$f(t) = t^{1/3}$$

Hint : Note that, $(\psi \circ f \circ \phi^{-1})(x) = x$ and $(\phi \circ \psi^{-1})(x) = x^{1/3}$. Thus $\phi \circ \psi^{-1}$ is of class C^{∞} but $\phi \circ \psi^{-1}$ is not of class C^{∞}. Again

$$(\psi \circ f \circ \phi^{-1})(x) = x$$

Also $f(y) = f(x)$ if and only if $y = x$. Thus f is one-one. Finally

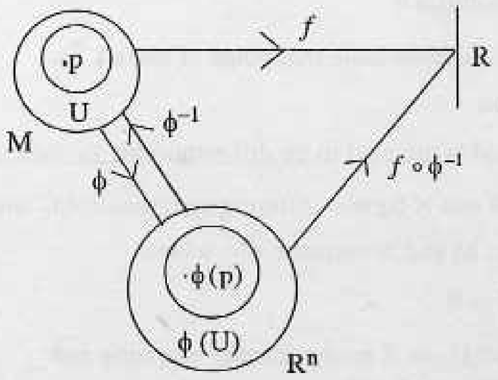
$f^{-1}(y) = y^3$, so that

$f(f^{-1}(y)) = y$ and $f^{1}(f(x)) = x$. Thus f is a bijection.

Note : A diffeomorphism f of M onto itself is called a <u>transformation</u> of M.

A real-valued function on M ; i.e.

$f : M \to R$



is said to be a <u>differentiable function of class</u> C^{∞}, if for every chart (U, ϕ) containing p of M, the function

3.4) $f \circ \phi^{-1} : \phi(U) \subset R^n \to R$

is of class C^{∞}.

We shall often denote by F(M), the set of all differentiable functions on M and will sometimes denote by F(p), the set of functions on M which are differentiable at p of M.

It is to be noted that such F(M) is

i) an algebra over R

ii) a ring over R

iii) an associative algebra over R and

iv) a module over R

Where the defining relations are

a) $(f+g)(p) = f(p) + g(p)$

b) $(fg)(p) = f(p)g(p)$

c) $(\lambda f)(p) = \lambda f(p), \quad \forall f, g \in F(M), \ \lambda \in R, \ p \in M.$

§ 1.4. Differentiable Curve :

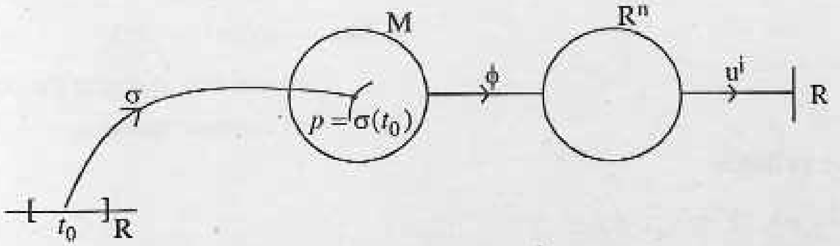
We are now in a position to define a curve on a manifold.

A differentiable curve through p in M of class C^r is a differentiable mapping

$\sigma : [a,b] \subset R \to M$, namely the restriction of a differentiable mapping of class C^r of an open interval] c, d [containing [a, b].

such that

4.1) $\sigma(t_0) = p \quad , a \leq t_0 \leq b$



Also

4.2) $(x^i \circ \sigma)(t) = (u^i \circ \phi)(\sigma(t)) = u^i(\phi(\sigma(t))) = u^i(\sigma^1(t), \cdots, \sigma^n(t)) = \sigma^i(t)$

We write it as

4.3) $x^i(t) = \sigma^i(t)$

The tangent vector to the curve $\sigma(t)$ at p is a function

$$X_p : F(p) \to R$$

defined by

4.4) $X_p f = \left[\frac{d}{dt} f(\sigma(t))\right]_{t=t_0} = \left[\lim_{h \to 0} \frac{f(\sigma(t+h) - f(\sigma(t))}{h}\right]_{t=t_0}$

where $\quad p = \sigma(t_0), f \in F(p)$

It can be shown that it satisfies

4.5) $X_p(af + bg) = a(X_p f) + b(X_p g)$: Linearity

4.6) $X_p(fg) = g(p)X_p f + f(p)X_p g, \ f, g \in F(p)$: Leibnitz Product Rule.

Note : Each function $X_p : F(p) \to R$, cannot be a tangent vector to some curve at $p \in M$, unless it is a linear function and satisfies Leibnitz Product Rule.

Exercises : 1. Let a curve σ on R^n be given by

$$\sigma^i = a^i + b^i t, \ i = 1, 2, \cdots, n$$

Find the tangent vector to the curve σ at the point (a^i).

2. If C is a constant function on M and X is a tangent vector to some curve σ at $p \in M$, then $X_p.C = 0$

[**Ans.** i) $(b^1, b^2, \cdots, b^n)$

ii) use 4.5), 4.6) and the definition of constant function.

Let us define

4.7) $(X_p + Y_p)f = X_p f + Y_p f$

4.8) $(bX_p) = bX_p f \quad , b \in R$

If we denote the set of tangent vectors to M at p by $T_p(M)$, then from 4.7) and 4.8) it is easy to verify that $T_p(M)$ is a vector space over R. We are now going to determine the basis of such vector space.

For each $i = 1, \ldots, n$, we define a mapping

$$\frac{\partial}{\partial x^i} : F(p) \to R$$

by

4.9) $\left(\frac{\partial}{\partial x^i}\right)_p f = \left(\frac{\partial f}{\partial x^i(t)}\right)(p)$

Note that

$$\left(\frac{\partial}{\partial x^i}\right)_p (af+bg) = \left(\frac{\partial (af+bg)}{\partial x^i(t)}\right)(p) \qquad \text{by } 4.9) \;, \qquad a, b \ \mathrm{R}, f, g \in \mathrm{F}\,(p)$$

$$= \left(a\frac{\partial f}{\partial x^i(t)}\right)(p) + \left(b\frac{\partial g}{\partial x^i(t)}\right)(p) \quad \text{by a) of 1.3}$$

$$= a\left(\frac{\partial f}{\partial x^i(t)}\right)(p) + b\left(\frac{\partial g}{\partial x^i(t)}\right) \quad \text{by a) of 1.3}$$

$$= a\left(\frac{\partial}{\partial x^i}\right)_p f + b\left(\frac{\partial}{\partial x^i}\right) g$$

Thus such a mapping satisfies linearity property. It can be shown that

$$\left(\frac{\partial}{\partial x^i}\right)_p (fg) = g(p)\left(\frac{\partial}{\partial x^i}\right)_p f + f(p)\left(\frac{\partial}{\partial x^i}\right)_p g$$

Let us define a differentiable curve

$\sigma : [a, b] \subset \mathrm{R} \to \mathrm{M}$

by

4.10) $\begin{cases} \sigma^i(t) = \sigma^i(t_0) + t \ \text{, for fixed } i \\ \sigma^j(t) = 0, \ j = 1, 2, \cdots, i-1, i+1, \cdots, n \end{cases}$

then

$$\left[\frac{d}{dt} f(\sigma(t))\right]_{t=t_0} = \left\{\sum_{i=1}^{n} \frac{\partial f(\sigma(t))}{\partial \sigma^i(t)} \cdot \frac{d\sigma^i(t)}{dt}\right\}_{t=t_0} \quad \text{by chain rule}$$

$$= \left(\frac{\partial f}{\partial x^i(t)}\right)_{\sigma(t_0)} \quad \text{for fixed } i, \text{ by (4.3)}$$

$$= \frac{\partial f}{\partial x^i(t)}(p)$$

$$= \left(\frac{\partial}{\partial x^i}\right)_p f \text{ by } (4.9)$$

Thus we can claim that each $\left(\frac{\partial}{\partial x^i}\right)$, $i = 1, 2, \cdots, n$ is a tangent vector to the curve σ defined above, at $p = \sigma(t_0)$.

Again from the definition of the tangent vector,

$$X_p f = \frac{d}{dt} f(\sigma(t))|_{t=t_0}$$

$$= \left\{\sum_{i=1}^{n} \frac{\partial f(\sigma(t))}{\partial \sigma^i(t)} \cdot \frac{d\sigma^i(t)}{dt}\right\}_{t=t_0} \qquad \text{by chain rule}$$

$$= \sum_{i=1}^{n} \left(\frac{dx^i(t)}{dt}\right)_{t=t_0} \frac{\partial f(\sigma(t_0))}{\partial x^i(t)} \qquad \text{by } (4.3)$$

$$= \sum_{i=1}^{n} \left(\frac{dx^i(t)}{dt}\right)_{t=t_0} \left(\frac{\partial}{\partial x^i(t)}\right)_p f$$

We write it as

4.11) $\quad X_p = \sum_{i=1}^{n} \xi^i(p) \left(\frac{\partial}{\partial x^i}\right)_p$ where

4.12) $\quad \xi^i(p) = \left(\frac{dx^i(t)}{dt}\right)_{t=t_0}, \quad i = 1, \cdots, n$

Thus each $\xi^i : \mathrm{M} \to \mathrm{R}$, $i = 1, \cdots, n$ is a differentiable function and every tangent vector, say X_p, to some curve, say $\sigma(t)$ at $p = \sigma(t_0)$ can be expressed as a linear combination of the tangent vector $\frac{\partial}{\partial x^i(t)}$, $i = 1, \cdots, n$ to the curve σ defined in (4.10)

If possible, for a given linear combination of the form $\sum \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)$, where ξ^i,s are functions on M, let us define a curve σ by

$$\sigma : \sigma^i(t) = \sigma^i(t_0) + \xi^i(p)t \text{ , } a \le t_0 \le b$$

then it can be shown that the tangent vector to this curve is $\sum \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)_p$

If we assume that

$$\sum \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)_p = 0$$

then,

$$\sum_i \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)_p x^k = 0 \text{ where } x^k : \mathrm{M} \to \mathrm{R},\ K = 1,2,\cdots n.$$

or $\sum_i \xi^i(p)\left(\frac{\partial x^k}{\partial x^i}\right)_p = 0$

$\therefore\ \xi^k(p) = 0.$ for $k = 1,2,\cdots n.$

Thus the set $\left\{\left(\frac{\partial}{\partial x^i}\right)_p : i = 1,\cdots,n\right\}$ is linearly independent. Hence we state

Theorem 1 : If $(x^1,\cdots,x^n)$ is a local coordinate system in a neighbourhood U of $p \in$ M, then, the basis of the tangent space $T_p(M)$ is given by

$$\left\{\left(\frac{\partial}{\partial x^1}\right)_p,\cdots,\left(\frac{\partial}{\partial x^n}\right)_p\right\}$$

Let us define $T(M) = \bigcup_{p \in M} T_p(M)$. This T(M) is called the tangent space of M.

§ 1.5. Vector Field :

In classical notation, if to each point p of R^3 or in a domain U of R^3, a vector

$\alpha : p \to \alpha(p)$

is specified, then, we say that a vector field is given on R^3 or in a domain U of R^3.

A vector field X on M is a correspondance that associates to each point $p \in M$, a vector $X_p \in T_p(M)$. In fact, if $f \in F(M)$, then Xf is defined to be a real-valued function on M, defined as follows

5.1) $(Xf)(p) = X_p f$

A vector field X is called differentiable if Xf is so for every $f \in F(M)$. Using (4.11) of § 1.4, a vector field X may be expressed as

5.2) $X = \sum \xi^i \frac{\partial}{\partial x^i}$

where ξ^i 's are differentiable functions on M.

Let $\chi(M)$ denote the set of all differentiable vector fields on M. We define

5.3) $\begin{cases} (X+Y)f = Xf + Yf \\ (bX)f = b(Xf) \end{cases}$

It is easy to verify that $\chi(M)$ is a vector space over R.

Also, for every $f \in F(M)$, fX is defined to be a vector field on M, defined as

5.4) $(fX)(p) = f(p)X_p$

Let us define a mapping as $[\,,\,] : F(M) \to F(M)$ as

5.5) $[X, Y]f = X(Yf) - Y(Xf), \ \forall X, Y \in \chi(M)$

Such a bracket is known as **Lie bracket** of X, Y.

Exercises : 1. Show that for every X, Y, Z in $\chi(M)$, for every f, g in F(M),

i) $[X, Y] \in \chi(M)$

ii) $[bX, Y] = [X, bY] = b[X, Y], b \in R$

iii) $[X + Y, Z] = [X, Z] + [Y, Z]$

iv) $[X, Y + Z] = [X, Y] + [X, Z]$

v) $[X, X] = \theta$ vi) $[X, Y] = -[Y, X]$

vii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=\theta$; Jacobi Identity

viii) $[fX, gY] = (fg)[X, Y] + \{f(Xg)\}Y - \{g(Yf)X\}$

a) $[X, fY] = f[X, Y] + (Xf)Y$

b) $[fX, Y] = f[X, Y] - (Yf)X$

2. In terms of a local co-ordinate system

i) $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

ii) $[X, Y] = \sum_{i,j}\left(\xi^i \frac{\partial \xi^j}{\partial x^i} - \zeta^i \frac{\partial \xi^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$, where $X = \xi^i \frac{\partial}{\partial x^i}$, $Y = \xi^j \frac{\partial}{\partial x^j}$

3. Complete [X, Y] where

i) $X = \frac{\partial}{\partial x^1}$, $Y = \frac{\partial}{\partial x^2} + e^{x1} \frac{\partial}{\partial x^3}$

ii) $X = x^1 x^2 \frac{\partial}{\partial x^1}$, $Y = x^2 \frac{\partial}{\partial x^2}$

4. Prove that

i) $\chi(M)$ is a F(M) module

Hints : 1. viii) Note that

$\{f(Yh)\}(p) = f(p)(Yh)_p$ by (5.4) of § 1.5)

$= f(p)\, Y_p{}^h$ by (5.1) of § 1.5)

Again, $\{(fY)\}(p) = (fY)(p)\, h$ by (5.1)

$= f(p)\, Y_p\, h$ by (5.4)

Thus $\{f(Yh)\}(p) = \{(fY)h\}(p), \forall p$

$f(Yh) = (fY)h$

Use the above result, 5.5) of § 1.5 & (4.6) of § 1.4, the result follows after a few steps.

§. 1.6. Integral Curve :

In this article, we are going to give the geometrical interpretation of a vector field.

Let Y be a vector field on M. The assignment of the vector Y_p

at each point $p \in U \subset M$, is given by

$Y : p \to Y_p \in T_p(M)$

A curve σ is an <u>integral curve</u> of Y if the range of σ is contained in U and for every $a \le t_0 \le b$ in the domain [a, b] of σ, the tangent vector to σ at $\sigma(t_0) = p$ coincides with Y_p i.e.

$$Y_p = Y_{\sigma(t_0)}$$

$$Y_p f = Y_{\sigma(t_0)} f, \qquad \forall f \in F(M)$$

$$= \left[\frac{d}{dt}(f \circ \sigma)(t)\right]_{t=t_0} \qquad \text{by (4.4) of § 1.4}$$

Using 4.11) § 1.4 one can write

$$\sum_i \eta^i(p)\left(\frac{\partial}{\partial x^i}\right)_p f = \left[\frac{d}{dt}(f \circ \sigma)(t)\right]_{t=t_0} \quad \text{where } \eta^i\text{'s are functions on M.}$$

$$= \sum \left(\frac{dx^i(t)}{dt}\right)_{t=t_0} \left(\frac{\partial}{\partial x^i}\right)_p f$$

As $\left\{ \frac{\partial}{\partial x^i} : i = 1, \cdots, n \right\}$ are linearly independent, we must have

$$\eta^i(p) = \left(\frac{dx^i}{dt}\right)_{t=t_0}$$

or $$\eta^i(\sigma(t))_{t=t_0} = \left(\frac{dx^i}{dt}\right)_{t=t_0}$$

or $$\eta^i(\sigma^1(t), \sigma^2(t), \cdots, \sigma^n(t))_{t=t_0} = \left(\frac{dx^i}{dt}\right)_{t=t_0}$$

Using (4.3) of § 1.4 we get

$$\eta^i\,(x^1(t), x^2(t), \cdots, x^n(t))_{t=t_0} = \left(\frac{dx^i}{dt}\right)_{t=t_0}$$

Hence they are related by

6.1) $\dfrac{dx^i}{dt} = \eta^i\big((x^1(t), \cdots, x^n(t)\big)$

Exercises : 1. Find the integral curve of a zero vector.

2. Find the integral curve of the following vector field

i) $X = x^1 \dfrac{\partial}{\partial x^1} + x^2 \dfrac{\partial}{\partial x^2}$ on R^2

ii) $X = e^{-x^1} \dfrac{\partial}{\partial x^1}$ on R

iii) $X = \dfrac{\partial}{\partial x^1} + (x^1)^2 \dfrac{\partial}{\partial x^2}$ on R^2

Solution : 2.i) From (6.1) of § 1.6, we see that

$$\frac{dx^1}{dt} = x^1, \quad \frac{dx^2}{dt} = x^2$$

or $\dfrac{dx^1}{x^1} = dt, \quad \dfrac{dx^2}{x^2} = dt$

Integrating

$\log x^1 = t + C \quad , \log x^2 = t + D$ say, where C, D are integrating constant.

When t = 0, if $x^1 = p^1$, $x^2 = p^2$, then from

$x^1 = Ce^t$ and $x^2 = De^t$

we find that

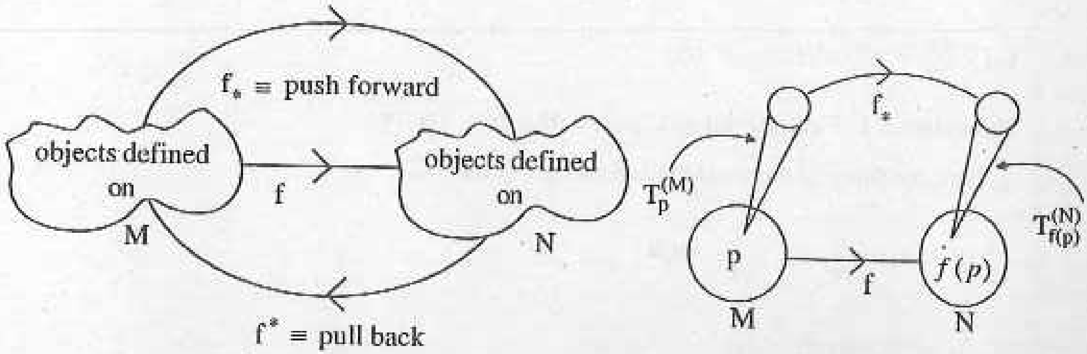
$p^1 = C, p^2 = D$

Thus $\gamma : (p^1e^t, \ p^2e^t)$ is the integral curve of X passing through the point (p^1, p^2)

§. 1.7 Differential of a mapping :

Let

f : M → N



be a a differentiable mapping of an n-dimensional manifold M to an m-dimensional manifold N. Let F(p) denote the set of all differentiable functions at $p \in M$ and $F(f(p))$ denote the set of all differentiable functions at $f(p) \in N$. Such a map f, induces a map

$f^*: F(f(p)) \to F(p)$, usually called pull back map.

and is defined by

7.1) $f^*(g) = g \circ f, \quad g \in F(f(p))$

called the <u>pull back</u> of g by f, which satisfies

7.2) $\begin{cases} f^*(ag + bh) = a(f^*g) + b(f^*h) \\ f^*(gh) = f^*(g) f^*(h) \end{cases}$ where $g, h \in F(f(p))$ and $a, b \in R$

The map f, also induces a linear mapping

$f_*: T_p(M) \to T_{f(p)}(N)$

such that

7.3) $\left(f_*(X_p)\right) g = X_p(g \circ f) = X_p\left(f^*(g)\right)$

called the push forward of X by f. Such f_* is also called derived linear map or Jacobian map or differential map of f on $T_p(M)$

Let us write

7.4) $f_*(X_p) = (f_*X)_{f(p)}$

We can also define push forward of X by f, geometrically, in the following manner :

Given a differential mapping

$$f : M \to N,$$

the differential of f at $p \in M$ is the linear mapping

$$f_* : T_p(M) \to T_{f(p)}(N)$$

defined as follows :

For each $X_p \in T_p$ (M), we choose a curve $\sigma(t)$ in M such that X_p is the tangent vector to the curve $\sigma(t)$ at $p = \sigma(t_0)$. Then $f_*(X_p)$ is defined to be the tangent vector to the curve $f(\sigma(t))$ at $f(p) = f(\sigma(t_0))$

Exercises :

1. If f is a differentiable map from a manifold M into another manifold N and g is a differentiable map from N into another manifold L, then, show that

i) $(g \circ f)_* = g_* \circ f_*$ ii) $(g \circ f)^* = f^* \circ g^*$

2. If f is a transformation of M and g is a differentiable function on M, prove that

i) $f_*[X,Y] = f_*[X,Y]$

ii) $f^*((f_*X)g) = X(f^*g)$

iii) $f_*(gX) = (g \circ f^{-1})(f_*X)$

for all vector fields X, Y on M.

Solution : 1. By definition, $f_*(X_p)$ is the tangent vector to the curve $f(\sigma(t))$ at $f(p) = f(\sigma(t_0))$ where X_p is the tangent vector to the curve $\sigma(t)$ at $p = \sigma(t_0)$. Hence by (4.4) of § 1.4

$$(f_*(X_p))g = \left[\frac{d}{dt}g(f(\sigma(t)))\right]_{t=t_0} \quad g \in F(f(p))$$

$$= \left[\frac{d}{dt}(g \circ f)(\sigma(t))\right]_{t=t_0}$$

$$= X_p \ (g \circ f) \text{ by 4.4) of § 1.4}$$

Hints 3. Given that

$$f : M \to M$$

is a transformation and hence for every $p \in M$, $f(p) = q$, say.

Thus, $p = f^{-1}(q)$

consequently, from 7.3) of § 1.7, we find that

$$\{(f_*(X_p)g\}f(p) = \{X_p(g \circ f)\}(p), \ \forall p \in M$$

or $\{(f_*(X_p)g)\}(q) = \{X_p(g \circ f)\}f^{-1}(q)$

or $(f_*(X))g = (X(g \circ f))f^{-1}$

Using this relation, one can deduce the three results.

We are now going to give a matrix representation of the linear mapping f_*.

Theorem 1 : If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N, where $(x^1, \cdots, x^n)$ is the local co-ordinate system in a neighbourhood of a point p of M and $(y^1, \cdots y^m)$ is the local co-ordinate system in a neighbourhood of $f(p)$ of N, then

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m} \frac{\partial f^j}{\partial x^i}_p \left(\frac{\partial}{\partial y^j}\right)_{f(p)} \text{ where } f^j = y^j \circ f$$

Proof : We write

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m} a_i^j \left(\frac{\partial}{\partial y^j}\right)_{f(p)}, \ i =, \ldots, n$$

where a_i^j's are unknown to be determined

or $$\left\{f_*\left(\frac{\partial}{\partial x^i}\right)\right\}y^k = \sum_{j=1}^{m} a_i^j \left(\frac{\partial}{\partial y^j}\right)_{f(p)} y^k$$ where each $y^k = \in F(f(p))$ $k = 1,\ldots,m$

using 7.3) of § 1.7, we find

$$\left(\frac{\partial}{\partial x^i}\right)_p (y^k \circ f) = \sum_{j=1}^{m} a_i^j \delta_j^k$$

or $$\left(\frac{\partial}{\partial x^i}\right)_p f^k = a_i^k$$

or $$\left(\frac{\partial f^k}{\partial x^i}\right)_p = a_i^k$$ by (4.9) of § 1.4

Thus

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m}\left(\frac{\partial f^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_{f(p)}$$

Note : 1. The matrix of f_*, denoted by (f_*) is given by

$$(f_*) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

Note : 2. The kernel of f_* is the set of $X_p \in T_p(M)$ for which

$$f_*(X_p) = \theta$$

The image of f_* is the set of $Y_{f(p)} \in T_{f(p)}(N)$ for which, there exists $X_p \in T_p(M)$ such that

$$f_*(X_p) = Y_{f(p)}$$

Now from a known theorem

dim (kernel f_*) + dim (Range f_*) = dim $T_p(M)$.

We write it as

7.5) dim (kernel f_*) + dim (Range f_*) = dim $T_p(M)$ for each $p \in M$

The dim (Range f_*) is called the rank f_*

If rank f_* = dim $T_p(M)$ we say

i) f is an immersion if dim M $\leq$ dim N and f(M) is an immersed submanifold of N

ii) f is an imbedding if f is one to one and an immersion and then f(M) is an imbedded submanifold of N

iii) f is a submersion if dim M $\geq$ dim N.

Exercises : 1. Show that

$$f: R \to R^2$$

given by

$$f(t) = (a \cos t, \sin t)$$

is an immersion.

2. Find (f_*) in the following cases

i) $f : R^2 \to R^2$ given by $f = \left((x^1)^2 + 2(x^2)^2, 3x^1x^2\right)$

ii) $f : R^2 \to R^2$ given by $f = \left(x^1e^{x^2} + x^2, x^1e^{x^2} - x^2\right)$ at (0, 0)

where (x^1, x^2) are the local co-ordinates on R^2

§. 1.8 f-related vector Field :

Let X and Y be fields on M and N respectively.

Then, for $p \in M$, let $X_p \in T_p(M)$ and $Y_{f(p)} \in T_{f(p)}(N)$ and such that

8.1) $$f_*(X_p) = Y_{f(p)}$$

where $f : M \to N$ is a differentiable mapping and f_* is already defined in the previous article. In such a case, we say that the two vector fields X, Y are f-related.

For $g \in F(f(p))$ we see that

$$\{f_*(X_p)\}g = Y_{f(p)}{}^g$$

Using 7.3) of § 1.7 and (5.1) of § 1.5 we find that

$$X_p(g \circ f) = (Yg)f(p), \ \forall p$$

Then

8.2) $\quad X(g \circ f) = (Yg)f$

If f is a transformation on M and

$$f_*(X_p) = X_{f(p)}$$

we say that, X is f-related to itself or X is invariant under f. We also write it as

8.3) $\quad f_* X = X$

Exercises : 1. Let $X_i, Y_i (i = 1, 2)$ be two f-related vector fields on M and N respectively. Show that the vector fields $[X_1, X_2]$ and $[Y_1, Y_2]$ are also f-related.

2. Prove that two vector fields X, Y respectively on M and N are f-related if and only if

$$f^*((f_* X)g) = X(f^* g)$$

where $f : M \to N$ is a C^∞ map.

3. If f is a transformation on M, show that, for every $X \in \mathfrak{X}(M)$, there exists a unique f-related vector field to X.

Solution : 1. From the definition of the Lie bracket, we see that

$$\begin{aligned} [X_1, X_2](g \cdot f) &= X_1(X_2(g \cdot f)) - X_2(X_1(g \cdot f)) \\ &= X_1((Y_2 g)f) - X_2((Y_1 g)f) \qquad \text{by (8.2) above} \\ &= \{Y_1(Y_2 g)\}f - \{Y_2(Y_1 g)\}f \qquad \text{by (8.2) above} \\ &= \{Y_1(Y_2 g) - Y_2(Y_1 g)\}f \end{aligned}$$

$[X_1, X_2](g \circ f) = \{[Y_1, Y_2]g\}f$ from the definition of the Lie Bracket. Hence from 8.2), one claims that $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.

§. 1.9 One parameter group of transformations on a manifold :

Definitioin

Let a mapping

$$\phi : R \times M \to M$$

is defined by

$$\phi : (t, p) \to \phi_t(p)$$

which satisfy

i) for each $t \in R$, $\phi(t, p) = \phi_t(p)$ is a transformation on M and $\phi_0(p) = p$

ii) for all t, s, t + s $\in$ R

$$\phi_t(\phi_s(p)) = (\phi_t \circ \phi_s)(p) = \phi_{t+s}(p)$$

Then the family $\{\phi_t | t \in R\}$ of mappings is called a one-parameter group of transformations on M.

Exercise : 1. Let $\{\phi_t | t \in R\}$ be a one-parameter group of mappings on M. Show that

i) $\phi_{-t} = (\phi_t)^{-1}$

ii) $\{\phi_t | t \in R\}$ form an abelion group.

Let us set

9.1) $\Psi(t) = \phi_t(p)$

Then $\Psi(t)$ is a differentiable curve on M such that

$$\Psi(0) = \phi_0(p) = p \quad \text{by Def. (i) above}$$

Such a curve is called the <u>orbit</u> through p of M. The tangent vector, say X_p to the curve $\psi(t)$ at p is therefore

9.2) $X_p f = \left[\frac{d}{dt} f(\Psi(t))\right]_{t=0} = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}, \quad \forall f \in F(M)$

In this case, we say that $\{\phi_t | t \in R\}$ induces the vector field X and X is called the generator of $\{\phi_t\}$. The curve $\Psi(t)$ defined by 9.1) is called the integral curve of X.

Exercises : 2. Show that the mapping

$$\phi : R \times R^3 \to R^3$$

defined by

$$\phi(t, p) = (p^1 + t,\ p^2 + t,\ p^3 + t)$$

is a one-parameter group of transformations on M and the generator is given by $\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}$

3. Let $M = R^2$ and let

$$\phi : R \times M \to M$$

be defined by

$$\phi(t, (x, y)) = \left(xe^{2t}, ye^{-3t}\right)$$

Show that ϕ defines a one-parameter group of transformation on R^2 and find its generator.

Note : Since every 1-parameter group of transformations induces a vector field on M, the question now arises whether every vector field on M generates one parameter group of transformations. This question has been answered in the negative.

Example : Let

$$X = -e^{x^1} \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$$

on $M = R^2$. Then,

$$\frac{dx^1}{dt} = -e^{x^1}, \quad \frac{dx^2}{dt} = 1$$

Thus $e^{-x^1} = t + A,\ x^2 = t + B$, where A, B are integrating constant.

Let $x^1 = p^1,\ x^2 = p^2$ for $t = 0$ Then, $A = e^{-p^1},\ B = p^2$.

Consequently the integral curve of X is $\psi(t)=\left(\log\frac{1}{t+e^{-p^1}}, t+p^2\right)$

which is not defined for all values of t in R. Thus, if $\psi(t)=\phi_t(p)$, then, X does not generate one parameter group of transformations.

Problem 7 leads us to the following definition :

Let I_ϵ be an open interval $(-\epsilon,\epsilon)$ and U be a nbd of p of M. Let a mapping

$$\phi: I_\epsilon \times U \to \phi_t(U) \subset M$$

denoted by

$$\phi(t,p)=\phi_t(p)$$

be such that

i) U is an open set of M

ii) for each $t\in I_\epsilon$, $\phi(t,p)\to\phi_t(p)$ is a transformation of U onto an open set $\phi_t(U)$ of M and $\phi_0(p)=p$

iii) if t, s, t + s are in I_ϵ and if $\phi_s(p)\subset$ U

$$\phi_t(\phi_s(p))=\phi_{t+s}(p)$$

Such a family $\{\phi_t | t\in I_\epsilon\}$ of mappings is called a local one parameter group of transformations, defined on $I_\epsilon \times U$.

We are now going to establish the following theorem

Theorem 1 : Let X be a vector field on a manifold M. Then, X generates a local one-parameter group of transformations in a neighbourhood of a point of M.

Proof : Let $(x^1,x^2,....x^n)$ be a local coordinate system in a neighbourhood U of p of M such that $\phi(p)=(0,...,0)\in R^n$, where (U, ϕ) is the chart at p of M. Then $x^i(p)=(u^i\circ\phi)(p)=0,\ i=1,...,n$

Let

$$X=\sum_i \xi^i(x^1,...,x^n)\frac{\partial}{\partial x^i}$$

be a given vector field on U, the neighbourhood of $p \in M$, where each ξ^{i}, s the components of X, are differentiable functioins on U of M. Then, for every X on M, we have a ϕ-related vector field on, $\phi(U) = U_1 C R^n$ with $\phi(p) = (0,...,0) \in U_1 C R^n$.

Let η^{i}, s be the components of the ϕ-related vector field on U_1 of R^n. Then by the existence theorem of ordinary differential equations, for each $\phi(p) \in U_1 \subset R^n$, there exists a $\delta_1 > 0$ and a neighbourhood V_1 of $\phi(p)$, $V_1 \subset U_1$ such that, for each $q = (q^1,...q^n) \in V_1$, $q = \phi(r)$, say, there exists n-tuple of $\overset{\infty}{C}$ functions $f^1(t,q),...f^n(t,q)$ defined on $I_{\delta_1} \subset I_{\epsilon_1}$ and mapping $f^i : I_{\delta_1} \to V_1 \subset U_1$, $i = 1,...,n$ which satisfies the system of first order differential equations

1) $\dfrac{df^i(t)}{dt} = \eta^i(t, \phi(p)), \quad i = 1,...,n$

with the initial condition

2) $f^i(0,q) = q^i$

Let us write

3) $\theta_t(q) = (f^1(t,q),...,f^n(t,q))$

We are going to show

$\theta_{t+s}(q) = \theta_t(\theta_s(q))$.

Note that if t, s, t + s are in I_{δ_1} and if $\theta_s(q) \in V_1 \subset U_1$ then each $f^i(t+s,q)$, $f^i(t,\theta_s(q))$ are defined on $I_{\delta_1} \times U_1$. Now let us set

$(g^1(t),...,g^n(t)) = (f^1(t+s,q),...f^n(t+s,q))$

For simplicity, we write

$(g^i(t)) = (f^i(t+s,q))$

Then each $g^i(t)$ is defined on $I_{\delta_1} \times U_1$ and hence satisfies 1) with the initial condition

4) $(g^i(o)) = (f^i(s,q))$

Also, let us set

$$\left(h^1(t),\ldots,h^n(t)\right)=\left(f^1(t,\phi_s(q)),\ldots,f^n(t,\theta_s(q)\right)$$

For simplicity, we write

$$\left(h^i(t)\right)=\left(f^i(t,\theta_s(q)\right)$$

then each $h^i(t)$ is defined on $I_{\delta_1}\times U_1$ and hence satisfies 1) with the initial condition

$$\left(h^i(o)\right)=\left(f^i(o,\theta_s(q))\right)$$

$$=\left(\theta_s(q)\right)^i \quad \text{by 2)}$$

$$=\left(f^i(s,q)\right) \quad \text{by 3)}$$

$$=\left(g^i(o)\right) \quad \text{by 4)}$$

Hence from the uniqueness we must have

$$\left(g^i(t)\right)=\left(h^i(t)\right)$$

Using 3) we must have

$$\theta_{t+s}(q)=\theta_t\left(\theta_s(q)\right).$$

Thus, we claim that, for every vector field defined in a neighbourhood U_1 of ϕ (p) of R^n, there exists $\left\{\phi_t \mid t\in I_{\delta_1}\right\}$ as its local 1-parameter group of transformations defined on $I_{\delta_1}\times U_1$.

Let us set

$$V=\phi^{-1}(V_1)\subset U$$

and define

$$\psi: I_\epsilon\times V\to\psi_t(V)\subset M$$

as follows

$$\psi_t(r)=\phi^{-1}\left(\theta(t,q)\right)$$

Then

i) $\psi(t,r)\to\psi_t(r)$ is a transformation of V onto $\psi_t(V)$ of M

ii) if t, s, t + s are in I_ϵ and if $\psi_s(r) \subset V$, then

$$\psi_t(\psi_s(r)) = \phi^{-1}(\theta(t, \phi(\psi_s(r)))$$

$$= \phi^{-1}(\theta(t+s, q))\text{, after a few steps}$$

$$= \psi_{t+s}(r)$$

Thus for the given vector field X, defined in a neighbourhood U of p of M, there exists $\{ \psi_t \mid t \in I_\epsilon \}$ as its local 1-parameter group of transformations, defined on $I_\epsilon \times V \subset U$ of M. Note that if we define

$$\gamma(t) = \psi_t(r) = \phi^{-1}(\phi(t, q)), \quad q = \phi(r)$$

$$= \phi^{-1}(\sigma(t)), \text{ say,}$$

then $\phi^{-1}(\sigma(t))$ is the integral curve of X.

This completes the proof.

Theorem 2 : Let ϕ be a transformation of M. If a vector field X generates ϕ_t as its local 1-parameter group of transformations, then, the vector field $\phi_* X$ will generate $\phi \phi_t \phi^{-1}$ as its local 1-parameter group of transformations.

Proof : Left to the reader.

Exercise : 4. Show that a vector field X on M is invariant under a transformation ϕ on M if and only if

$$\phi \circ \phi_t = \phi_t \circ \phi$$

where ϕ_t is the local 1-parameter group of transformations induced by X.

We now give a geometrical interpretation of [X, Y], for every vector field X, Y on M.

Theorme 3 : If X generates ϕ_t as its local 1-parameter group of transformations, then, for every vector field Y on M.

$$[X, Y]_q = \lim_{t \to 0} \frac{1}{t}\left\{ Y_q - ((\phi_t)_* Y)_q \right\} \text{ where } q = \phi_t(p) \text{ and } (\phi_t)_* Y_p = ((\phi_t)_* Y)\phi_t(p)$$

To prove the theorem, we require some lemmas which are stated below :

Lemma 1 : If $\psi(t, p)$ is a function on $I_\epsilon \times M$, where I_ϵ is an open interval $(-\epsilon, \epsilon)$ such that

$$\psi(0, p) = 0, \quad \forall p \in M$$

then, there exists a function $h(t, p)$ on $I_\epsilon \times M$ such that

$$t\,h(t, p) = \psi(t, p)$$

Moreover

$$h(o, p) = \psi'(o, p), \text{ Where } \psi' = \frac{d\psi}{dt}.$$

Proof : It is sufficient to define

$$h(t, p) = \int_0^1 \psi'(ts, p)\frac{d(ts)}{t}$$

Hence by the fundamental theorem of calculus

$$h(t, p) = \left[\frac{1}{t}\psi(ts, p)\right]_0^1$$

$$\therefore\ th(t, p) = \psi(t, p)$$

Also from above

$$h(o, p) = \int_0^1 \psi'(o, p)ds \quad = \psi'(o, p)\,[s]_0^1 = \psi'(o, p)$$

Lemma 2 : If f is a function on M and X is a vector field on M which induces a local 1-parameter group of transformations ϕ_t then there exists a function g_t defined on $I_\epsilon \times V$, V being the neighbourhood of p of M, where

$$g_t(p) = g(t, p)$$

such that

$$f(\phi_t(p)) = f(p) + tg_t(p)$$

Moreover,

$$X_p f = g(o,p) = g_0(p)$$

Symbolically,

$$Xf = g_0 \text{ on M.}$$

Proof : Let us set

$$\tilde{f}(t,p) = f(\phi_t(p)) - f(\phi_0(p)), \quad \forall\, p \in M$$

Then $\tilde{f}(t,p)$ is a function on $I_\varepsilon \times M$ such that

$$\tilde{f}(o,p) = f(\phi_0(p)) - f(\phi_0(p)) = 0, \quad \forall\, p \in M$$

Hence by Lemma 1, there exists a function, say, g(t, p) on $I_\varepsilon \times V$, $V \subset M$ being the neighbourhood of p of M, such that

$$tg(t,p) = \tilde{f}(t,p)$$

$$\therefore \quad g(t,p) = \frac{f(\phi_t(p)) - f(\phi_0(p))}{t}$$

or, $$g(o,p) = \lim_{t \to 0} \frac{1}{t}\{f(\phi_t(p))\} - f(\phi_0(p)) = X_p f$$

As,

$$tg(t,p) = f(\phi_t(p)) - f(p)$$

we find that

$$f \circ \phi_t = f + tg_t$$

Proof of the main theorem :

Let us write

$$\phi_t(p) = q$$

$$\therefore \quad p = \phi_t^{-1}(q) = \phi_{-t}(q)$$

Now,

$$\{((\phi_t)_* Y)f\}(q) = \{Y(f \circ \phi_t)\}(p) = \{Y(f + tg_t)\}(p) \text{ by Lemma 2}$$

or $\quad (Yf)(q) - (((\phi_t)_* Y))(q) = (Yf)(q) - (Yf)(p) - t(Yg_t)(\phi_{-t}(q))$

or, $\quad \left(\lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\}\right)f = \lim_{t \to 0} \frac{(Yf)(q) - (Yf)(p)}{t} = \lim_{t \to 0} (Yg_t)(\phi_{-t}(q))$

$$= \lim_{t \to 0} \frac{1}{t}\{(Yf)(q) - (Yf)(p)\} - (Yg_0)(q)$$

$$= \lim_{t \to 0} \frac{1}{t}\{(Yf)(q) - (Yf)(p)\} - y_q(Xf), \text{ by Lemma 2}$$

From the definition we find that,

$$X_q f = \lim_{t \to 0} \frac{1}{t}\{f(\phi_t(q)) - f(q)\}$$

or $\quad -X_q f = \lim_{t \to 0} \frac{1}{t}\{f(p) - f(q)\}$

Taking $f = Yf$, we find from above after a few steps

$$X_q(Yf) = \lim_{t \to 0} \frac{1}{t}\{(Yf)(q) - (Yf)(p)\}$$

Thus we write,

$$\left(\lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\}\right)f = X_q(Yf) - Y_q(Xf) = \{[X, Y]f\}(q), \text{ after a few steps.}$$

$$[X, Y]_q = \lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\}$$

Note : We abbreviate the above result as

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}\{Y - ((\phi_t)_* Y)\}$$

Corollary : 1. Show that

$$(\phi_s)_*[X, Y] = \lim_{t \to 0} \frac{1}{t}\{(\phi_s)_* Y - ((\phi_{s+t})_* Y)\}$$

Proof : From the last theorem

$$(\phi_s)_*[X,Y] = \lim_{t\to 0}\frac{1}{t}\left\{(\phi_s)_*Y - (\phi_s)_*(\phi_t)_*Y\right\}, \text{ as } (\phi_s)_* \text{ is a linear mapping}$$

$$\lim_{t\to 0}\frac{1}{t}\left\{(\phi_s)_*Y - ((\phi_s\circ\phi_t)_*Y)\right\}, \text{ from a known result}$$

Using the definition of local 1-parameter group of transformations, the result follows immediately.

Corollary 2 : Show that

$$(\phi_s)_*[X.Y] = -\left(\frac{d((\phi_t)_*Y)}{dt}\right)_{t=s}$$

Proof : Left to the reader

Corollary 3 : Let X, Y generate ϕ_t and ψ_s respectively, as its local 1-parameter group of transformations. Then

$$\phi_t\circ\psi_s = \psi_s\circ\phi_t$$

if and only if [X, Y].

Proof : Let

$$\phi_t\circ\psi_s = \psi_s\circ\phi_t$$

Then from Exercise 4, the vector field Y is invariant under ϕ_t. Hence by § 1.8

$$(\phi_t)_*Y = Y$$

Consequently from Theorem 3, [X, Y] = 0

Converse result follows from corollary 2.

A vector field X on a manifold M is said to be <u>complete</u> if it induces a one parameter group of transformations on M.

Theorem 4 : Every vector field on a compact manifold M is complete.

Proof : Let X be a given vector field on M. Then by Theorem 1, X induces $\{\phi_t\}$ as its local 1-parameter group of transformations in a neighbourhood U of p of M and $t \in I_\epsilon \subset R$. If p runs over M, then for each p, we get a neighbourhood U(p) and I_ϵ (p), where all such U(p) from an open coverings of M. Since M is compact, every open covering {U(p)} of M has a finite subcovering $\{ U(p_i) : i = 1, ..., n \}$ say. If we let

$$\in = \min\{\in(p_1), \in(p_2), \ ..., \ \in(p_n)\}$$

then, there is a t such that for $|t| < \in$

$$\phi_t(p) : (-\in, \in) \times M \to M$$

is local 1-parameter group of transformations on M. We are left to prove that $\phi_t(p)$ is defined on R × M.

Case a) : $t \geq \in$

We write

$$t = k \cdot \frac{\in}{2} + r \quad , |r| < \frac{\in}{2}, \text{ k being integer}$$

Then $\phi_t = \phi_{k\frac{\in}{2}+r}$

$$= \phi_{k\frac{\in}{2}} \circ \phi_r$$

$$= \underbrace{\phi_{\frac{\in}{2}} \circ \phi_{\frac{\in}{2}} \ ..... \circ \phi_{\frac{\in}{2}}}_{\text{k times}} \cdot \phi_r$$

Similarly for $t \leq -\in$, we can show that

$$\phi_t = \phi_{-\frac{\in}{2}} \ .... \ \phi_{-\frac{\in}{2}} \cdot \phi_r$$

Thus ϕ_t is a local 1-parameter group of transformations on M.

Combining all the cases, we claim that ϕ_t is defined on R × M. Hence X induces ϕ_t as its 1-parameter group of transformations on a compact manifold M. Thus X is a complete vector field.

§ 1.10 Cotangent Space :

Note that χ (M) is a vector space over the field of real numbers. A mapping

$\omega : \chi(M) \to F(M)$

that satisfies

$\omega(X+Y) = \omega(X) + \omega(Y)$

$\omega(bX) = b\,\omega(X)$, $b \in R$ and for all X, Y $\in$ X(M),

is a linear mapping over R.

The linear mapping

$\omega : \chi(M) \to F(M)$

denoted by

$\omega : X \to \omega(X)$

is called a 1-form on M.

Let

$$D_1(M) = \{\omega, \mu, \ldots \mid \omega : \chi(M) \to F(M)\}$$

be the set of all 1-forms on M. Let us define

$$10.1)\quad \begin{cases} (\omega+\mu)(X) = \omega(X) + \mu(X) \\ (b\omega)(X) \quad = b\omega(X) \end{cases}$$

It can be shown that D_1 (M) is a vector space over R, called the dual of χ (M).

For every p $\in$ M, $\omega(X) \in F(M)$ is a mapping

$\omega(X) : M \to R$ defined by

$$10.2)\quad \{\omega(X)\}(p) = \omega_p(X_p)$$

so that

$$\omega_p : T_p(M) \to R$$

Thus $\omega_p \in$ dual of $T_p(M)$. We write the dual of $T_p(M)$ by $T_p^*(M)$ and is the cotangent space of $T_p(M)$. Elements of $T_p^*(M)$ are called the covectors at p of M or linear functionals on $T_p(M)$.

For every f $\in$ F(M), we denote the total differential of f by df and is defined as

10.3) $(df)_p(X_p) = (Xf)(p) = X_p f, \ \forall p$

We also write it as

10.4) (df) (X) = Xf

Exercises : 1. Show that for every f $\in$F(M), df is a 1-form on M.

2. If $(x^1, x^2, \ldots, x^n)$ are coordinate functions defined in a neighbourhood U of p $\in$ M, show that each $dx^i, i = 1, \ldots, n$ is a 1-form on U $\subset$ M.

Solution : 2 Note that

$(dx^i)(X+Y) = (X+Y)x^i$, (10.4) above

$= Xx^i + Yx^i$

$= (dx^i)(X) + (dx^i)(Y)$, by (10.4)

Similarly it can be shown that

$$(dx^i)(bX) = b(dx^i)(X)$$

Thus each $dx^i, i = 1, \ldots, n$ is a 1-form on R.

From Exercise 2 above, it is evident that each $(dx^i)_p \in T_p^*(M)$, for i =1, ..., n. We now define

10.5) $(dx^i)_p\left(\frac{\partial}{\partial x^j}\right)_p = \delta^i_j$

Let $\omega_p \in T^*_p(M)$ be such that

10.6) $\omega_p\left(\frac{\partial}{\partial x^j}\right)_p = (f_j)_p$ where each $(f_j)_p \in R$

If possible, let $\mu_p \in T^*_p(M)$ be such that

$$\mu_p = (f_1)_p(dx^1)_p + \ldots\ldots + (f_n)_p(dx^n)_p$$

then

$$\mu_p\left(\frac{\partial}{\partial x^1}\right)_p = \{(f_1)_p(dx^1) + \cdots + (f_n)_p(dx^n)_p\}\left(\frac{\partial}{\partial x^1}\right)_p = (f_1)_p \text{ by (10.5)}$$

Proceeding in this manner we will find that

$$\mu_p\left(\frac{\partial}{\partial x^1}\right)_p = (f_1)_p = \omega_p\left(\frac{\partial}{\partial x^i}\right) \text{ by (10.6)}$$

As $\left\{\frac{\partial}{\partial x^i} : i = 1, \ldots, n\right\}$ are linearly independent, we must have

$$\mu_p = \omega_p.$$

Thus any $\omega_p \in T^*_p(M)$ can be expressed uniquely as

10.7) $\omega_p = \sum (f_i)_p (dx^i)_p$

$\therefore T^*_p(M) = \text{span}\{(dx^1)_p, \ldots, (dx^n)_n\}$

Finally if

$\sum_i (f_i)_p (dx^i)_p = 0$, then,

$$\sum_i (f_i)_p (dx^i)_p \left(\frac{\partial}{\partial x^k}\right)_p = 0$$

i.e. $(f_k)_p = 0.$ by (10.5)

Similarly it can be shown that

$$(f_1)_p = ... = (f_n)_p = 0$$

Thus the set $(f_1)_p = ... = (f_n)_p = 0$ is linearly independent and we state

Theorem 1 : If $(x^{1},....x^{n})$ are local coordinate system in a neighbourhood U of p of M, then the linear functionals $\{(dx^i)_p := 1,....,n\}$ on $T_p(M)$ form a basis of $T_p^*(M)$.

Note that

$(dX^i)(X) = Xx^i$ by 10.4)

$= \sum \xi^j \frac{\partial}{\partial x^j} x^i$ by 5.2) of § 1.5

10.8) $(dx^i)(X) = \xi^i$

Thus, one can find

$$(df)(X) = Xf = \sum \xi^i \frac{\partial}{\partial x^i} f = \sum \frac{\partial f}{\partial x^i} dx^i(X) \text{ from above}$$

Hence we write

10.9) $\quad df = \sum \frac{\partial f}{\partial x^i} dx^i$

For every $\omega \in D_1$ (M), we define $f\omega$ to be a 1 form in M and write

10.10) $\quad (f\omega)(X) = f(\omega(X))$

Note : D_1(M) is a F(M)-module

§. 1.11 r-form, Exterior Product :

An r-form is a skew-symmetric mapping

$\omega : \chi(M)\times\cdots\cdots\times\chi(M)\to F(M)$

such that

i) w is R-linear

ii) if σ is a permutation of 1,2......r with

$(1, 2, \ldots\ldots, r)\to(\sigma(1), \sigma(2), \ldots \sigma(r))$ then

11.1) $\omega(X_1, X_2, \ldots\ldots, X_r) = \frac{1}{r!}\sum_{\sigma}(\text{sgn}\sigma)\,\omega(X_{\sigma(1)}, X_{\sigma(2)}, \ldots\ldots X_{\sigma(r)})$ where $(\text{sgn}\,\sigma)$ is + 1 or -1 according as σ is even or odd permutation .

If ω is a r-form and μ is a s-form, then, the exterior product or wedge product of ω and μ denoted by $\omega\wedge\mu$ is a (r+s)-form. defined as

11.2 $(\omega\wedge\mu)(X_1, X_2, \ldots\ldots X_r, X_{r+1}, \ldots X_s)$

$$= \frac{1}{(r+s)!}\sum_{\sigma}(\text{sgn}\,\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(r)})\,\mu\,(X_{\sigma(r+1)}, \ldots\ldots X_{\sigma(r+s)})$$

where σ ranges over the permutation (1, 2,......r+s), $X_i \in \chi(M). i = 1, 2, \ldots\ldots, r+s$

For convenience, we write

11.3) $f\wedge g = fg,\ f, g\in F(M).$

It can be shown that, if ω is a r-form

11.4)
$$\begin{cases}(f\wedge\omega)(X_1, \ldots, X_r) = f\omega(X_1, X_2, \ldots, X_r)\\ (\omega\wedge f)(X_1, \ldots, X_r) = f\omega(X_1, \ldots, X_r)\end{cases}$$

Again, if ω and μ are 1-forms, then

11.5) $(\omega\wedge\mu)(X_1, X_r) = \frac{1}{2}\{\omega(X_1)\mu(X_2) - \omega(X_2)\mu(X_1)\}$

The exterior product obeys the following properties :

11.6)
$$\begin{cases} \omega \wedge \mu = -\mu \wedge \omega, \quad \omega \wedge \omega = 0 \\ f\omega \wedge \mu = f(\omega \wedge \mu) = \omega \wedge f\mu \\ f\omega \wedge g\mu = fg\omega \wedge \mu \quad , \quad \omega \wedge \mu = (-1)^{rs} \mu \wedge \omega, \quad \omega : r\text{- form } \mu : s\text{- form} \\ (\omega + \mu) \wedge \gamma = \omega \wedge \gamma + \mu \wedge \gamma \end{cases}$$

Exercises : 1. If ω is a 1-form and μ is a 2-form, show that

$$(\omega \wedge \mu)(X_1, X_2, X_3) = \frac{1}{3}\{\omega(X_1)\mu(X_2, X_3) + \omega(X_2)\mu(X_3, X_1) + \omega(X_3)\mu(X_1, X_2)\}$$

2. Compute

i) $(2du^1 + du^2) \wedge (du^1 - du^2)$

ii) $(6du^1 \wedge du^2 + 27du^1 \wedge du^3) \wedge (du^1 + du^2 + du^3)$

Solution : 2 i) $(2du^1 + du^2) \wedge (du^1 - du^2)$

$= 2du^1 \wedge (du^1 - du^2) + du^2 \wedge (du^1 - du^2)$

$= -2du^1 \wedge du^2 + du^2 \wedge du^1$ as $du^i \wedge du^i = 0$

$= -3\, du^1 \wedge du^2$ by 11.6)

Theorem 1 : In terms of a local coordinate system $(x^1, x^2, ..., x^n)$ in a neighbourhood U of p of M, an r-form ω can be expressed uniquely as

11.7) $\omega = \sum_{i_1 < i_2 < ... < i_r} f_{i_1 i_2 ... i_r}\, dx^{i_1} \wedge dx^{i_2} ... \wedge dx^{i_r}$ where $f_{i_1 i_2 ... i_r}$ are differentiable functions on M.

Proof : Let $D_r(M)$ denote the set of all differentiable r-forms on M. In terms of a local co-ordinate system $(x^1, x^2, ..., x^n)$ in a neighbourhood U of p of M, the set $\{ dx^{i_1} \wedge ... \wedge dx^{i_r} : 1 \le i_1 < i_2 < ... < i_r \le n \}$ form a basis of $D_r(M)$. Using 11.2) we find

i) $(dx^{i_1} \wedge ... \wedge dx^{i_r})\ (X_1, X_2, ..., X_r) = \frac{1}{r!} \sum_{\sigma} (\text{sgn}\, \sigma)\, dx^{i_1}\left(X_{\sigma(1)}\right) ... dx^{i_r}\left(X_{\sigma(r)}\right)$

$i_1 < i_2 < ... < i_r$

where σ ranges over the permutation (1, 2, ..., r) and each $X_i \in \chi$ (M).

Let

ii) $\quad X_k = \sum_{j_m=1}^{n} \xi_k^{j_m} \frac{\partial}{\partial x^{j_m}}$

where ξ 's are functions, called the components of X_k.

Using ii), we get from i)

$$\left(dx^{i_1} \wedge \dots \wedge dx^{i_r}\right)(X_1, \dots, X_r) = \frac{1}{r!} \sum_\sigma (\text{sgn}\,\sigma)\, dx^{i_1}\left(\sum \xi_{\sigma(1)}^{j_m} \frac{\partial}{\partial x^{jm}}\right) \dots dx^{i_r}\left(\sum \xi_{\sigma(r)}^{j_s} \frac{\partial}{\partial x^{jk}}\right)$$

$$i_1 < i_2 < \dots < i_r$$

Using (10.5) of § 1.10, we get from above

iii) $\quad (dx^{i_1} \wedge \dots \wedge dx^{i_r})\,(X_1, X_2, \dots, X_r) = \frac{1}{r!} \sum_\sigma (\text{sgn}\,\sigma)\; \xi_{\sigma(1)}^{i_1} \dots \xi_{\sigma(r)}^{i_r} \; i_1 < i_2 < \dots < i_r$

Using ii) in (11.1) of § 1.11, , we find

$$\omega\,(X_1, X_2 \dots, X_r) = \frac{1}{r!} \sum_\sigma (\text{sgn}\,\sigma)\; \omega\left(\sum \xi_{\sigma(1)}^{j_m} \frac{\partial}{\partial x^{j_m}}, \dots, \sum \xi_{\sigma(r)}^{j_s} \frac{\partial}{\partial x^{j_s}}\right)$$

As each ω is R-linear, we find from above

$$= \frac{1}{r!} \sum_\sigma (\text{sgn}\,\sigma) \sum_{j_m, \dots, j_s} \xi_{\sigma(1)}^{j_m} \dots \xi_{\sigma(r)}^{j_s}\; \omega\left(\frac{\partial}{\partial x^{j_m}}, \dots, \frac{\partial}{\partial x^{j_s}}\right)$$

Changing the dummy indices $j_m \to i_1, \dots, j_s \to i_r$ we get

$$= \frac{1}{r!} \sum_\sigma (\text{sgn}\,\sigma) \sum_{i_1, \dots, i_r} \xi_{\sigma(1)}^{i_1} \dots \xi_{\sigma(r)}^{i_r}\; \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}\right)$$

Using iii) we find from above

$$= \sum_{\substack{i_1, \dots, i_r \\ i_1 < i_2 < \dots < i_r}} (dx^{i_1} \wedge \dots \wedge dx^{i_r})\,(X_1, X_2 \dots, X_r) f_{i_1 i_2 \dots i_r} \quad , \text{ where}$$

$$\omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}\right) = f_{i_1 i_2 \dots i_r}$$

Thus

$$\omega(X_1, X_2 ..., X_r) = \sum_{\substack{i_1,...,i_r \\ i_1<i_2<...<i_r}} f_{i_1 i_2 ... i_r} (dx^{i_1} \wedge ... \wedge dx^{i_r})(X_1,...,X_r), \ \forall X_1,...,X_r$$

Hence we can write

$$\omega = \sum_{i_1<i_2<...<i_r} f_{i_1 i_2 ... i_r} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_r}$$

This completes the proof.

Exercises : 3. Show that a set of 1-forms $\{\omega_1, \omega_2, ..., \omega_k\}$ is linearly dependent if and only if

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0$$

4. Let $\{\omega_1, \omega_2, ..., \omega_k\}$ be k-independent 1-forms on M. If μ_i be k 1-forms satisfying

$$\sum_i \mu_i \wedge \omega_i = 0$$

show that

$$\mu_i = \sum A_{ij} \, \omega_j \ \text{ with } A_{ij} = A_{ji}$$

Solution : 3. Let the given set of 1-forms be linearly dependent. Hence any one of them, say, ω_{k-1} can be expressed as a linear combination of the rest i.e.

$$\omega_{k-1} = b_1\omega_1 + b_2\omega_2 + \cdots + b_{k-2}\omega_{k-2} + b_k\omega_k, \text{ where each } b^i \in R$$

$$\therefore \quad \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_k$$

$$= \omega_1 \wedge \omega_2 \wedge \cdots \wedge (b_1\omega_1 + b_2\omega_2 + ... + b_{k-2}\omega_{k-2} + b_k\omega_k) \wedge \omega_k$$

$$= b_1\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_1 \wedge \omega_k + \cdots + b_k\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \wedge \omega_k$$

= 0 by 11.6) of this article.

Converse follows easily.

4. As $\{\omega_1, ..., \omega_k\}$ is a independent set of of 1-forms, we complete the basis of D_1 (M) by taking 1-forms $\omega_{k+1}, ..., \omega_n$. Thus the basis of D_1 (M) is given by $\{\omega_1, ..., \omega_k, \omega_{k+1}, ..., \omega_n\}$.

Consequently any 1-from μ_i, $i = 1,...k$ can be expressed as

i) $\mu_i = \sum_{m=1}^{k} A_{im}\omega_m + \sum_{p=k+1}^{n} B_{ip}\omega_p, \; i = 1, 2, \ldots k$

Given that

$$\sum_i \mu_i \wedge \omega_i = 0$$

i.e. $\mu_1 \wedge \omega_1 + \mu_2 \wedge \omega_2 + \cdots + \mu_k \wedge \omega_k = 0$

Using i) and 11.6) one gets after a few steps

$$\sum_{i<j\le k} (A_{ij} - A_{ji})\,\omega_i \wedge \omega_j + \sum_{\substack{i\le k \\ j>k}} B_{ji}\omega_i \wedge \omega_j = 0$$

As ω 's are given to be linearly independent, so we must have

$$A_{ij} - A_{ji} = 0 \text{ and } B_{ij} = 0$$

i.e. $A_{ij} = A_{ji}$

Consequently i) reduces to

$$\mu_i = \sum A_{ij}\omega_j \text{ with } A_{ij} = A_{ji}$$

§. 1.12. Exterior Differentiation :

The exterior derivative, denoted by d on D is defined as follows :

i) $d(D_r) \subset D_{r+1}$

ii) for $f \in D_0$, df is the total differential

iii) if $\omega \in D_r$, $\mu \in D_s$ then

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^r \omega \wedge d\mu$$

iv) $d^2 = 0$

From 11.7) of § 1.11 we find that

12.1) $d\omega = \sum_{i_1<i_2<\cdots<i_r} df_{i_1\cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}$

Exercises : 1. Find the exterior differential of

i) $x^2 y\, dy - xy^2\, dx$

ii) $\cos(xy^2)\, dx \wedge dz$

iii) $x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

2. Find the exterior differential of

$d\omega \wedge \mu - \omega \wedge d\mu$

A form ω is said to be <u>closed</u> if

12.2) $d\omega = 0$

If ω is a r–form and

12.3) $d\mu = \omega$

for some (r–1) form μ then, ω is said to be an <u>exact</u> form.

Exercise : 3. Test whether ω is closed or not where

i) $\omega = xy\, dx + \left(\frac{1}{2}x^2 - y\right) dy$

ii) $\omega = e^x \cos y\, dx + e^x \sin y\, dy$

Theorem 1 : If ω is a 1-form, then

$$d\omega(X_1, X_2) = \frac{1}{2}\{X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2])\}$$

Proof : Without any loss of generality, one may take an 1-form as

$$\omega = f\, dg,\ f, g \in F(M)$$

$\therefore \quad d\omega(X_1, X_2) = (df \wedge dg)(X_1, X_2)$

Using 11.5) of § 1.11, we find

$$d\omega(X_1, X_2) = \frac{1}{2}\{(df)(X_1)\, dg\,(X_2) - (df)(X_2)(dg)(X_1)\}$$

Using (10.4) of § 1.10, we get

$$d\omega(X_1,X_2) = \frac{1}{2}\{(X_1f)(X_2g)-(X_2f)(X_1g)\}$$

$$=\frac{1}{2}\{X_1(f(X_2g)-f(X_1(X_2g))-X_2(f(X_1g)+f(X_2(X_1g))\}$$ on

using (4.6) of § 1.4

Now $\omega(X_1)=(fdg)(X_1)=f(dg(X_1))$, as $(f\omega)(X)=f(\omega(X))$

$= f(X_1g)$ by (10.4) of § 1.10

by $\omega(X_2)=f(X_2g)$

Thus we get from above

$$d\omega(X_1,X_2)=\frac{1}{2}\left[X_1(\omega(X_2)-X_2(\omega(X_1)-f\{X_1(X_2g)-X_2(X_1g)\}\right]$$

$$=\frac{1}{2}\{X_1(\omega(X_2)-X_2(\omega(X_1)-f([X_1,X_2]g)\}$$

$$\therefore \quad d\omega(X_1,X_2)=\frac{1}{2}\{X_1(\omega(X_2))-X_2(\omega(X_1))-\omega([X_1,X_2])\}$$

This completes the proof.

Existence and Uniqueness of Exterior Differentiation :

Without any loss of generality we may take an r-form as

$\omega = f_{i_1 i_2 \dots i_r} dx^{i_1}\wedge\dots\wedge dx^{i_r}$, $f_{i_1\dots i_r}\in$ F(M)

Let us define an R-linear map

$d : D \to D$ as

12.4) $d\omega = df_{i_1 i_2 \dots i_r}\ dx^{i_1}\wedge\dots\wedge dx^{i_r}$

Clearly i) $d(D_r)\subset D_{r+1}$ and

ii) if ω is a 0-form, then $d\omega$ is the total differential of ω.

iii) Let $\mu\in D_s$ and it is enough to consider

$$\mu = g_{j_1 \dots j_s}\, dx^{j_1} \wedge \dots \wedge dx^{j_s} \quad , g_{j_1 \dots j_s} \in F(M)$$

then $d(\omega \wedge \mu) = d\left(f_{i_1 i_2 \dots i_r} g_{j_1 \dots j_s} dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}\right)$

Using 12.1 we get

$$d(\omega \wedge \mu) = d(f_{i_1 \dots i_r} g_{j_1 \dots j_s}) \wedge dx^{i_1} \wedge dx^{j_1} \wedge \dots \dots \wedge dx^{j_s}$$

$$= (g_{j_1 \dots j_s} df_{i_1 \dots i_r} + f_{i_1 \dots i_r} dg_{j_1 \dots j_s}) \wedge dx^{i_r} \wedge \dots \wedge dx^{j_1} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}$$

$$= g_{j_1 \dots j_s}\, df_{i_1 \dots i_r} \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s} + f_{i_1 \dots i_r}\, dg_{j_1 \dots j_s} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}$$

$$= df_{i_1 \dots i_r} \wedge dx^{i_1} \wedge dx^{i_r} \wedge g_{j_1 \dots j_s} dx^{j_1} \wedge \dots \dots \wedge dx^{j_s} + (-1)^r f_{i_1 \dots i_r} dx^{i_1} \wedge \dots \dots dx^{i_r} \wedge dg_{j_1 \dots j_s} \wedge dx^{j_1} \wedge dx^{j_s}$$

$$= d\omega \wedge \mu + (-1)^r \omega \wedge d\mu$$

iv) Again using (10.9) of § 1.10 in (12.4) we see that

$$d\omega = \sum_{i_k} \frac{\partial f}{\partial x^{i_k}} dx^{i_k} \wedge dx^{i_1} \wedge \dots \dots \wedge dx^{i_r}$$

or d $d(d\omega) = \sum_{i_s} \sum_{i_k} \frac{\partial^2 f}{\partial x^{i_s} \partial x^{i_k}} dx^{i_s} \wedge dx^{i_k} \wedge dx^{i_1} \wedge \dots \dots \wedge dx^{i_r}$

$$= 0,$$

If $i_s \neq i_k$, then, $dx^{i_s} \wedge dx^{i_k} = 0$

Thus existence of such d is established.

It is easy to establish the uniqueness of d.

Thus there exist a unique exterior differentiation on D.

§. 1.13 Pull-back Differential Form :

Let M be an n-dimensional and N be an m-dimensional manifold and

$f : M \to N$

be a differentiable mapping. Let $T_p(M)$ be the tangent space at p of M where $T^*_{f(p)}(N)$ is its dual. Let $T_{f(p)}(N)$ be the tangent space at f(p) of N where $T^*_{f(p)}(N)$ is its dual. If $(x^1,....x^n)$ and $(y^1,....y^m)$ are the local corrdinate system at p of M and at f(p) of N respectively, then, it is known that $\{\frac{\partial}{\partial x^i}: i=1,.....,n\}$ *and* $\{\frac{\partial}{\partial y^j}; j=1,.....,m\}$ are respectively the basis of $T_p(M)$ and $T_{f(p)}(N)$. Consequently $\{dx^i : i=1,...n\}$ and $\{dy^j : j = 1, ..., m\}$ are the basis of $T^*_p(M)$ and $T^*_{f(p)}(N)$ respectively.

Let ω be a 1-form on N. We define an 1-form on M, called the pull-back 1 form of ω on M, denoted by $f^*\omega$, as follows

13.1) $[f^*(\omega_{f(p)})](X_p) = (f^*\omega)_p(X_p) = \omega_{f(p)}(f_*(X_p), \forall$ p of M.

where f_*, $f^*$are already defined in § 1.7

So, we write

13.2) $f^*(\omega_{f(p)}) = (f^*\omega)_p$

then, by 7.4) of § 1.7, we get from 13.1, on using 13.2)

13.3) $(f^*\omega)_p(X_p) = \omega_{f(p)}(f_*X)_{f(p)}, \forall p \text{ of } M$

Therefore we may write, for a 1 form ω on N and a vector field X on M by

13.4) $(f^*\omega)(X) = \omega(f_*X)$

Theorem 1 : If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N, where $(x^1, x^2,....x^n)$ is the local coordimate system in a neighbourhood of a point p of M and $(y^1....y^m)$ is the local coordinate system in a neighbourhood of f(p) of N, then

$$f^*(dy^i)_{f(p)} = \sum_{i=1}^{n}(\frac{\partial f^j}{\partial x^i})_p (dx^i)_p \quad \text{where } f^j = y^j.f, \ i=1,....m$$

Proof : Since $f^*(dy^j)_{f(p)}$ is a co-vector at P on M, it can be expressed as the linear combination of the basis co-vectors $(dx^i)_p$ at P and we take

$$f^*(dy^j)_{f(p)} = \sum_{i=1}^{n} a_i^j (dx^i)_p$$

Where a_i^j's are unknown s to be determined

or $\{f^*(dy^j)_{f(p)}\}\left(\frac{\partial}{\partial x^k}\right)p = \sum_i a_i^j (dx^i)_p \left(\frac{\partial}{\partial x^k}\right)p$

usinng 10.5 of § 1.10 we find that

$$\left(f^*(dy^j)_{f(p)}\right)\left(\frac{\partial}{\partial x^k}\right)_p = a_i^j \,\delta_k^i = a_k^j \text{ for } (dx^i)_p\left(\frac{\partial}{\partial x^k}\right)_p = \frac{\partial x^i}{\partial k^k} = \delta_k^i$$

By (13.1). one reduces to

$$\left(dy^j\right)_{f(p)}\left\{f_*\left(\frac{\partial}{\partial x^k}\right)_p\right\} = a_k^j$$

using Theorem 1 of § 1.7 we find

$$\sum_{j=i}^{m}(dy^j)_{f(p)}\left(\frac{\partial f^s}{\partial x^k}\right)_p\left(\frac{\partial}{\partial y^s}\right)_{f(p)} = a_k^j$$

Using (10.5) of § (1.10) we find

$$\left(\frac{\partial f^j}{\partial x^k}\right)_p = a_k^j$$

Thus we get

$$f^*(dy^j)_{f(p)} = \sum_{i=1}^{n}\left(\frac{\partial f^j}{\partial x^k}\right)_p (dx^i)_p \quad , \; j = 1, \ldots, m; \; f^j = y^j \circ f$$

Note : 1. Using (10.9) of § 1.10, one find from above theorem

13.5) $f^*(dy^j)_{f(p)} = (df^j)_p, j = 1, \ldots. m$

we can also write it as

13.6) $f^*(dy^j)_{f(p)} = df^j = d(y^j . f)$

2. If ω is a 1-form, then, its pull-back 1-form $f^*\omega$ is given by

13.7) $f^*\omega = \sum_j \omega_j df^J$, where ω_j are the components of ω

(Prove it.)

Exercises : 1 If $f: M \to R^3$ be such that

$f(u,v) = (u\cos v, u\sin v, av)$ where

$x^1 = u\cos v, x^2 = u\sin v, x^3 = av$

then for a given 1-form ω, $\omega = x^1dx^1 - dx^2 + x^2dx^3 on\ R^3$, compute $f^*\omega$.

2. If $f: M \to R^3$ be such that

$f(u,v) = (a\cos u\ Sin v, a\ Sin u\ Sin v, a\ Cos v)$ then for a given 1-form ω

$\omega = dx^1 + adx^2 + dx^3$ on R^3, determine $f^*\omega$.

3. Let ω be the 1-form in $R^2 - \{o,o\}$ by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Let U be the set in the plane $(r,0)$ given by

$$U = \{r > 0; 0 < \theta < 2\pi\}$$

and let f : U→ R^2 be the map $f(r,\theta) = \begin{cases} x = r\ Cos\theta, \\ y = r\ Sin\theta \end{cases}$ compute $f^*\omega$

Let us now suppose that ω be a r-form on N. In the same manner, as defined earlier, we define an r-form on M, called the <u>pull-back r-form</u> on M, denoted by $f^*\omega$, as follows :

13.8) $\left(f^*\left(\omega_{f(p)}\right)\right)\left((X_1)_p, \ldots, (X_r)_p\right) = \omega_{f(p)}\left(f_*(X_1)_p, \ldots, f_*(X_r)_p\right), \forall p$

We also write it as

13.9) $(f^*\omega)(X_1 ..., X_r) = \omega(f_* X_1, ..., f_* X_r)$

Proposition : 1. Let

$$f : M^n \to N^m$$

be a map, ω and μ be r-forms on N and g be a 0-form on N. Then

a) $f^*(\omega + \mu) = f^*\omega + f^*\mu$

b) $f^*(g\omega) = f^*(g) f^*\omega$

Proof : a) As ω and μ are r-forms on N, $(\omega + \mu)$ is also so. Hence

$$\left(f^*(\omega+\mu)_{f(p)}\right)(X_1, X_2 ..., X_r) = (\omega+\mu)_{f(p)}(f_* X_1, ..., f_* X_r)$$

$$= \omega_{f(p)}(f_* X_1, ..., f_* X_r) + \mu_{f(p)}(f_* X_1, ..., f_* X_r)$$

$$= \left(f^*\left(\omega_{f(p)}\right)\right)(X_1, ..., X_r) + \left(f^*\left(\mu_{f(p)}\right)\right)(X_1, ..., X_r) \text{ by 13.8)}$$

$$\therefore\ f^*(\omega+\mu)_{f(p)} = f^*(\omega)_{f(p)} + f^*(\mu)_{f(p)},\quad \forall f(p)$$

Hence

$$f^*(\omega + \mu) = f^*\omega + f^*\mu$$

b) Note that if ω is a r-form and g is a o-form, then $g\omega$ is again a r-form. Using (13.8) one gets

$$\left(f^*(g\omega)_{f(p)}\right)(X_1, ..., X_r) = (g\omega)_{f(p)}(f_* X_1, f_2 X_2, ..., f_* X_r)$$

$$= \left(g(fp)\omega_{f(p)}\right)(f_* X_1, f_* X_2, ..., f_* X_r)$$

$$= \left((g \circ f)(p)\omega_{f(p)}\right)(f_* X_1, f_* X_2, ..., f_* X_r)$$

$$= (g \circ f)(p)\omega_{f(p)}(f_* X_1, ..., f_* X_r)$$

$$= \left(f^*(g)(p)(f_* \omega_{f(p)}\right)(f_* X_1, ..., f_* X_r)$$

or $\left(f^*(g\omega)_{f(p)}\right) = f^*(g)(p)\left(f^*(\omega)_{f(p)}\right)$

or $\left(f^*(g\omega)\right)_p = \left(f^*(g)\right)(p)\left(f^*(\omega)\right)_p$, $\forall$ p

Hence $f^*(g\omega) = f^*(g)f^*(\omega)$.

Exercises : 4. Show that

$$f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu$$

5. Prove that

$$(f \circ h)^*\omega = h^*(f^*\omega)$$

Note : From Theorem 1 of § 1.11, we see that, any r-form ω can be expressed as

$$\omega = \sum_{i_1 < i_2 < \dots < i_r} g_{i_1 i_2 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

where $g_{i_1 i_2 \dots i_r}$ are differentiable functions on N. Then

$$f^*\omega = \sum_{i_1 < i_2 < \dots < i_r} f^*\left(g_{i_1 i_2 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}\right)$$

$= \sum f^* g_{i_1 i_2 \dots i_r} f^*dx^{i_1} \wedge \dots \wedge f^*dx^{i_r}$ by the Proposition 1(b) and Exercise 4 above

$$= \sum \left(g_{i_1 \dots i_r} \circ f\right) f^*dx^{i_1} \wedge \dots \wedge f^*dx^{i_r}$$

Using 13.5) of § 1.13 we see that

13.10) $$f^*\omega = \sum_{i_1 < i_2 < \dots < i_r} \left(g_{i_1 \dots i_r} \circ f\right) df^{i_1} \wedge \dots \wedge df^{i_r}$$

Exercise : 7. Let M be a circle and M' be R^2 so that

$$f : M \to M'$$

be defined by

$$x^1 = r\cos\theta, \; x^2 = r\sin\theta$$

If $\omega = a\,dx^1 + b dx^2$ and $\mu = \frac{1}{a} dx^1 + \frac{1}{b} dx^2$, find $f^*(\omega \wedge \mu)$

Solution : In this case,

$$\omega_1 = a,\ \omega_2 = b,\ \mu_1 = \frac{1}{a},\ \mu_2 = \frac{1}{b}$$

$$df^1 = \cos\theta dr - r\sin\theta\, d\theta$$

$$df^2 = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$\therefore \qquad f^*\omega = a(\cos\theta\, dr - r\sin\theta\, d\theta) + b(\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= (a\cos\theta + b\sin\theta)\, dr + (br\cos\theta - ar\sin\theta)\, d\theta$$

and $f^*\mu = \frac{1}{a}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{1}{b}(\sin\theta\, dr + r\cos\theta\, d\theta)$

$$= \left(\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta\right)dr + \left(\frac{r}{b}\cos\theta - \frac{r}{a}\sin\theta\right)d\theta$$

Using Exercise 5, one finds that

$$f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu$$

$$= \{(a\cos\theta + b\,Sin\theta)dr + (b\,r\cos\theta - a\,r\sin\theta)\, d\theta\}$$

$$\wedge \left\{\left(\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta\right)dr + \left(\frac{r}{b}\cos\theta - \frac{r}{a}\sin\theta\right)d\theta\right\}$$

$$= (a\cos\theta + b\sin\theta)\left(\frac{r}{b}\cos\theta - \frac{r}{a}\sin\theta\right)\ dr \wedge d\theta +$$

$$+ (b\,r\cos\theta - a\,r\sin\theta)\left(\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta\right)\ d\theta \wedge dr$$

$$= r\left(\frac{a}{b} - \frac{b}{a}\right) dr \wedge d\theta \ \text{ where } d\theta \wedge dr = -dr \wedge d\theta.$$

Theorem 2 : For any form ω,

$$d(f^*\omega) = f^*(d\omega)$$

where the symbols have their usual meanings.

Proof : We shall consider the following cases.

i) ω is a o-form

ii) ω is a r-form

Case i) : In this case, let $\omega = h$, where h is a differentiable function

Then $\{f^*(dh)\}(X) = dh(f_*X)$

$= (f_*X)h$ by (10.4) of § 1.10

$= X(h \circ f)$ by (7.3) of § 1.7

$= d(h \circ f)(X)$ by (10.4) of § 1.10

$= \{d(f^*h)\}(X)$ by (10.4) of § 1.10

or $f^*(dh) = d(f^*h)$

The result is true in this case.

Case ii) : In this case, we assume that the result is true for $(r-1)$ form. Without any loss of generality, we may take an r-form ω as

$$\omega = g_{i_1 i_2 \ldots i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

$$\text{or } f^*\omega = f^*\left(g_{i_1 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}\right)$$

$$= f^*\left(g_{i_1 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}\right)$$

$$= f^*\left(g_{i_1 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_{r-1}}\right) \wedge f^*(dx^{i_r})$$

$$\text{or } d(f^*\omega) = d\left\{f^*\left(g_{i_1 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_{r-1}}\right) \wedge f^*(dx^{i_r})\right\}$$

Using (12.1) of § 1.12 we find that

$$d(f^*\omega) = d\left\{f^*\left(g_{i_1 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_{r-1}}\right) \wedge f^*(dx^{i_r})\right\} +$$

$$+(-1)^{r-1} f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right) \wedge d\left(f^*(dx^{i_r})\right)$$

Note that dx^{i_r} is a 1-form and hence the theorem is true in this case. Thus

$$d\left(f^*(dx^{i_r})\right) = f^*\left(d(dx^{i_r})\right) = 0 \text{ by (12 1) of § 1.12}$$

Hence

$$d(f^*\omega) = d\left\{f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right)\right\} \wedge f^*(dx^{i_r})$$

$$= f^*\left\{d\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right)\right\} \wedge f^*(dx^{i_r}) \text{ , as}$$

the result is true for $(r-1)$ form

$$= f^*\left\{\left(dg_{i_1 \dots i_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right)\right\} \wedge f^*(dx^{i_r}) \text{ by (12.1) of § 1.12}$$

$$= f^*\left(dg_{i_1 \dots i_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}} \wedge dx^{i_r}\right) \text{ by known result}$$

Thus $\qquad d(f^*\omega) = f^*(d\omega)$

and hence the result is true for r-form also.

Combining we claim that

$$d(f^*\omega) = f^*(d\omega)$$

i.e. d and f commute each other.

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UNIT - 2

§. 2.1 Lie group, Left translation, Right translation :

Let G be a differentiable manifold. If G is a group and if the map

$$(g_1, g_2) \to g_1 g_2$$

from G × G to G and the map

$$g \to g^{-1}$$

from G to G are both differentiable, then G is called a Lie group.

Exmaple : Let GL(n, R) denote the set of all nonsingular n × n matrices over real numbers. GL(n, R) is a group under matrix multiplication. Define

$$\phi(A) = (a_{11}, a_{12}, \ldots, a_{1n}; a_{21}, a_{22}, \ldots, a_{2n}; \ldots; a_{n1}, a_{n2}, \ldots, a_{nn})$$

then

$$\phi : GL(n, R) \to R^{n^2}$$

is a mapping of class C^{∞}. Hence GL(n, R) is a Lie group.

Note : Lie groups are the fundamental building blocks for gauge theories.

For every a ∈ G, a mapping

$$L_a : G \to G$$

defined by

2.1) $L_a x = ax, \quad \forall\, x \in G$

is called a **Left translation** on G.

Similarly, a mapping

$$R_a : G \to G$$

defined by

2.2) $R_a x = xa, \quad \forall\, x \in G$

is called a **right translation** on G.

Note that

$$L_a L_b x = L_a(bx) = abx \text{ and } L_{ab} x = abx$$

$$\therefore L_a L_a = L_{ab}$$

$$R_a R_b x = R_a(bx) = xba \text{ and } R_{ab} x = xba$$

$$\therefore R_a R_b = R_{ba}$$

$$L_a R_b x = L_a(xb) = axb \text{ and } R_b L_a x = R_b(ax) = axb$$

$$\therefore L_a R_b = R_b L_a$$

Thus

2.3) $L_a L_b = L_{ab}, \; R_a R_b = R_{ba}, \; L_a R_b = R_b L_a$

Again

$L_b L_a x = L_b(ax) = bax \neq abx \neq L_a L_b x$, Thus

2.4) $L_b L_a \neq L_a L_b$, unless G is commutative

Taking $b = a^{-1}$ in 2.3) we find

$$L_a L_{a^{-1}} = L_{aa^{-1}} \text{ by 2.3)}$$

$$= L_e$$

Thus

2.5) $L_{a^{-1}} = (L_a)^{-1}$

It is evident that, for every a $\in$ G, each L_a and R_a are diffeomorphism on G.

Exercise : 1 Show that the set of all left (right) translation on G form a group.

2. Let $\phi : G_1 \rightarrow G_2$ be a homeomorphism of a Lie group G_1 to another Lie group G_2. Show that

i) $\phi \circ L_a = L_{\phi(a)} \circ \phi$

ii) $\phi \circ L_b = R_{\phi(b)} \circ \phi, \quad \forall \, a, b$ in G.

3. Let ϕ be a 1-1 non-identity map from G to G. If

$$\phi \circ L_g = L_g \circ \phi$$

is satisfied for all g $\in$ G, then there is a h $\in$ G such that $\phi = R_h$.

Solution : 2. From the definition of group homeomorphism of a Lie group G_1 to another Lie group G_2,

$$\phi(ab) = \phi(a)\phi(b), \ \forall a, b \text{ in } G_1$$

i) $(\phi \circ L_a)x = \phi(L_a x) = \phi(ax) = \phi(a)\phi(x) = L_{\phi(a)}\, \phi(x) = (L_{\phi(a)} \circ \phi)x, \ \forall x \text{ in } G_1$

$$\therefore \ \phi \circ L_a = L_{\phi(a)} \circ \phi$$

Similarly ii) can be proved.

3. As G is a group, e $\in$ G (identity). Further ϕ is a 1–1 map from G to G, so for e $\in$ G, there is h in G such that

$$\phi(e) = h$$

Note that

$\phi(e) \neq e$, because, ϕ is not an identity map.

Now for g $\in$ G,

$$g = ge$$

$$\therefore \ \phi(g) = \phi(ge)$$

$$= \phi(L_g e)$$

$$= (\phi \circ L_g)(e)$$

$$= (L_g \circ \phi)(e), \text{ as given}$$

$$= Lg(\phi(e))$$

$$= Lgh$$

$$= gh$$

$$= R_h g$$

$$\therefore \ \phi = R_h, \forall g$$

§. 2.2. Invariant Vector Field :

We have already defined a vector field to be invariant under a transformation in § 1.8. Note that, in a Lie group G, for every a, b in G, each L_a, R_b is a transformation on G. Thus we can define invariant vector field under L_a, R_b.

A vector field X on a Lie group G is called a **left invariant** vector field on G if

2.6) $(L_a)_* X_p = X_{L_a(p)}$, $\forall p \in G$, where $(La)_*$ is the differential of L_a.

Thus from § 1.7

$$\left((L_a)_* X_p\right)_{L_a(p)} = X_{L_a(p)}$$

We write it as

2.7) $(L_a)_* X = X$

Similarly for a right invariant vector field, write

2.8) $(R_a)_* X = X$

From § 1.7) we know that

$$\left((L_a)_* X_p\right) g = X_p(g \circ L_a)$$

or $\left((L_a)_* X_p\right)_{L_a(p)} g = X_p(g \circ L_a)$

If $L_a(p) = q$ then $p = (L_a)^{-1} q = L_{a^{-1}} q = a^{-1} q$

Thus the above relation reduces to

2.9) $\left((L_a)_* X\right)_q g = X_{a^{-1}q}(g \circ L_a)$

Let g be the set of all left invariant vector field on G.

If X, Y, $\in$ g, a, b $\in$ R, then

2.10) $(L_p)_*(aX + bY) = a(L_p)_* X + b(L_p)_* Y = aX + bY$, $(L_p)_*$ being linear explained in Unit 1.

2.11) $(L_p)_*[X, Y] = \left[(L_a)_* X, (L_p)_* Y\right]$, see § 1.7 = [X, Y]

Thus $aX + bY \in g$ and $[X, Y] \in g$. Consequently g is a vector space over R and also a Lie-algebra. The Lie algebra formed by the set of all left invariant vector fields on G is called the **Lie algebra of the Lie group G.**

Note that every left invariant vector field is a vector field i.e.

$$g \subset \chi(G)$$

where $\chi(G)$ denotes the set of all vector field on G. The converse is not necessarily true.

The converse will be true if a condition is satisfied by a vector field. The following theorem states such condition.

Theorem 1 : A vector field X on a Lie group G is left invariant if and only if for every $f \in F(G)$

2.12) $\quad (Xf) \circ L_a = X(f \circ L_a)$

Proof : Let X be a left invariant vector field on a Lie group G. Then for every $f \in F(G)$, we have from (2.6)

$$\{(L_a)_* X_p\} f = X_{L_a(p)} f$$

or $\quad X_p(f \circ L_a) = (Xf) L_a(p) \quad$ by Q 1.7

or $\quad \{X(f \circ L_a)\}(p) = (Xf \circ L_a)(p) \quad , \forall p \in G$

$$\therefore \; Xf \circ L_a = X\,(f \circ L_a)$$

Conversely let (2.12) be true. Reversing the steps one gets the desired result.

Note : i) The behaviour of a Lie group is determined largely by its behaviour in the neighbourhood of the identity element e of G. The behaviour can be represented by an algebraic structure on the tangent space of e, called the **Lie algebra** of the group.

ii) Note that, two vector spaces U and V are said to be isomorphic, if a mapping

$$f : \mathrm{U} \rightarrow \mathrm{V}$$

is i) linear and ii) has an inverse $f^{-1} : \mathrm{V} \rightarrow \mathrm{U}$

Theorem 2 : As a vector space, the Lie subalgebra g of the Lie group G is isomorphic to the tangent space $T_e(G)$ at the identity element e $\in$ G.

Proof : Let us define a mapping

$\phi : g \to T_e$ (G) by

i) $\phi(X) = X_e$

Note that, for every X, Y in g, $X + Y \in g$ and

$$\phi(X+Y) = (X+Y)_e \quad \text{by i)}$$
$$= X_e + Y_e$$
$$= \phi(X) + \phi(Y)$$

Also for $b \in R$, $bX \in g$ and

$$\phi(bX) = (bX)_e \quad \text{by i)}$$
$$= bX_e$$
$$= bX \quad \text{by i)}$$

Thus ϕ is linear.

We choose $X_a \in T_a(G)$ such that

ii) $(L_a)_* V_e = X_a$, , Where $V_e \in T_e(G)$.

Then $(L_s)_* X_{s^{-1}a} = (L_s)_*(L_{s^{-1}a})_* \mathrm{V}_e$ from above

$$= (L_s \circ L_{s^{-1}a})_* \mathrm{V}_e \text{ from § 1.7}$$
$$= (L_{ss^{-1}}a)_* \mathrm{V}_e \text{ by (2.3)}$$
$$= (L_a)_* \mathrm{V}_e$$
$$= X_a \quad \text{, as chosen}$$

or $\quad ((L_s)_* X)_{L_s(s^{-1}a)} = X_{L_s(s^{-1}a)}$ by Q 1.7

or $\quad (L_s)_* X = X$

$\therefore \; X \in g$

We define

$\phi^{-1} : \mathrm{T_e(G)} \to g$ by

iii) $\phi^{-1}(V_e) = X$

Then $(\phi\phi^{-1})V_e = \phi(\phi^{-1}(V_e)) = \phi(X) = X_e$ ii), where $(L_e)_*$ is the identity differential on G.

or $(\phi\phi^{-1})V_e = V_e$

Further, $(\phi^{-1}\phi)X = \phi^{-1}(\phi(X)) = \phi^{-1}(X_e)$, by i)

$= \phi^{-1}((L_e)_* V_e)$ by ii)

$= \phi^{-1}(V_e)$

$= X$ by iii)

Thus an inverse mapping exists and we claim that

$g \cong T_e(G)$

Exercises : 1. If, X, Y are left invariant vector fields, show that [X, Y] is also so.

2. If c_{ij}^k $(i, j, k = 1, 2, ..., n)$ are structure constants on a Lie group G with respect to the basis $\{X_1, X_2, ..., X_n\}$ of g, show that

i) $c_{ij}^k = -c_{ji}^k$

ii) $c_{ij}^k c_{ks}^l + c_{js}^k c_{ki}^l + c_{si}^k c_{kj}^l = 0$

Solution : 1. From Q 1.7), we see that

$\{(L_a)_*[X, Y]\}f = [X, Y](f \circ L_a)$

$= X(Y(f \circ L_a)) - Y(X(f \circ L_a))$, from the definition of Lie Bracket

$= X\{((L_a)_* Y)f\} - Y\{((L_a)_* X)f\}$ by § 1.7

$= X(Yf) - Y(Xf)$ by (2.7)

$= [X, Y]f$ from the definition of Lie Bracket

$\therefore \ (L_a)_*[X, Y] = [X, Y], \forall f$

Using (2.7), we see that [X, Y] is a left invariant vector field.

2. Using problem 1 above, we see that every $[X_i, X_j] \in g$ as $X_i \in g$, $i = 1, \ldots, n$.

Since $\{X_1, X_2, \ldots, X_n\}$ is a basis of g, every $[X_i, X_j] \in g$ can be expressed uniquely as,

1) $[X_i, X_j] = c_{ij}^k X_k$ where $c_{ij}^k \in$ R

i) Note that if $i = j, [X_i, X_j] = 0$

So, let $i \neq j$. Then from a known result,

$$[X_i, X_j] = -[X_j, X_i]$$

Using 1) we find that

$$c_{ij}^k X_k = -c_{ji}^k X_k$$

As the set $\{X_1, \ldots, X_n\}$ is a basis of g and hence linearly independent, we must have

$$c_{ij}^k = -c_{ji}^k$$

ii) Using Jacobi Identity, we find that

$$\left[[X_i, X_j], X_s\right] + \left[[X_j, X_s], X_i\right] + \left[[X_s, X_i], X_j\right] = \theta$$

Hence from 1)

$$c_{ij}^k [X_k, X_s] + c_{js}^k [X_k, X_i] + c_{si}^k [X_k, X_j] = \theta \text{ as } [bX, Y] = b[X, Y], b \in R$$

Again applying 1), we find that

$$c_{ij}^k c_{ks}^t X_t + c_{js}^k c_{ki}^t X_t + c_{si}^k c_{kj}^t X_t = \theta$$

As $\{X_1, \ldots, X_n\}$ is a basis and hence linearly independent, we must have

$$c_{ij}^k c_{ks}^t + c_{js}^k c_{ki}^t + c_{si}^k c_{kj}^t = 0$$

§. 2.3 Invariant Differential Form :

A differential form ω on a Lie group G is said to be left invariant if

2.13) $L_a^*(\omega_{L_a(p)}) = \omega_p$, $\forall p \in G$

we write it as

2.14) $L_a^* \omega = \omega$ and call $L_a^* \omega$, the pull-back differential form of ω.

Similarly, a differential form ω on a Lie group G is said to be right invariant if

2.15) $R_a^* \omega = \omega$

A differential form, which is both left and right invariant, is called a biinvariant differential form.

Exercises : 1. If ω_1, ω_2 are left invariant differential forms, show that, each $d\omega, \omega_1 \wedge \omega_2$ is also so.

2. Prove that a differential 1-form ω on a Lie group is left invariant if and only if for every left invariant vector field X on G, $\omega(X)$ is a constant function on G.

3. Let $\phi : G \to G$ be such that $\phi(a) = a^{-1}$, $\forall a \in G$. Show that a form ω is left invariant if and only if $\phi^* \omega$ is right invariant.

4. Prove that the set of all left invariant forms on G is an algebra over R. Such a set is denoted by A, say.

5. If g^* denotes the dual space of g, then, prove that

$$A \cong g^*$$

where A is the set already defined in Exercise 4 above.

Solution : 1. From Q 1.13, we see that

$$L_a^* (d\omega_1) = d(L_a^* \omega_1)$$

where $L_a^* \omega_1$ is the pull-back 1 form of ω_1

Using on (2.14) on the right hand side of the above equation, we see that

$$L_a^* (d\omega_1) = d\omega_1$$

Consequently, $d\omega_1$ is a left invariant differential form.

It can be proved easily that $\omega_1 \wedge \omega_2$ is a left invariant differential form.

2. Let us consider a differential 1-form ω. Then for every $a \in G$, $L_a^* \omega$ will be defined as the pull-back differential 1-form. Consequently from the definition of pull-back,

$$\left(L_a^* \omega_{L_a(p)}\right)(X_p) = \omega_{L_a(p)}\left((L_a)_* X_p\right), \ \forall\, p \in G$$

Let us consider X to be left invariant. Then on using (2.6) on the right hand side of the above equation, we get

1) $$\left(L_a^* \omega_{L_a(p)}\right)(X_p) = \omega_{L_a(p)}\left(X_{L_a(p)}\right)$$

Let us now consider ω to be left invariant 1-form. Then by (2.13), we get from 1)

$$\omega_p(X_p) = \omega_{L_a(p)}\left(X_{L_a(p)}\right)$$

$$= \omega_{ap}(X_{ap})$$

Taking $p = e$, we see that

$$\omega_e(X_e) = \omega_{ae}(X_{ae}) = \omega_a(X_a)$$

Consequently, $\omega(X)$ is a constant function on G.

Conversely, if $\omega(X)$ is a constant function on G, then

$$\omega_p(X_p) = \omega_{ap}(X_{ap})$$

Hence 1) reduces to

$$\left(L_a^* \omega_{L_a(p)}\right) X_p = \omega_p(X_p)$$

or $$L_a^* \omega_{L_a(p)} = \omega_p$$ which is (2.13)

Thus ω is a left invariant differential form.

This completes the proof.

Theorem 1 : If g is a Lie subalgebra of a Lie group G and g^* denotes the set of all left invariant form on G, then

$$d\omega(X,Y) = -\frac{1}{2}\omega([X,Y]) \text{ where } \omega \in g^*, \quad X, Y, \in g$$

Note : Such an equatioin is called Maurer-Carter Equation.

Proof : From theorem 1 of § 1.12, we know that

$$d\omega(X,Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(Y)) - \omega([X,Y])\} \text{ for every vector field X, Y}$$

If X, Y are in g then by Exercise 2, $\omega(X), \omega(Y)$ are constant functions on G. Hence by Exercise 2 of § 1.4),

$$X.\omega(Y) = 0, \quad Y.\omega(X) = 0$$

Thus the above equation reduces to

$$d\omega(X,Y) = \frac{1}{2}\omega([X,Y]).$$

Exercise : 6. Show that

$$d\omega^i = -\frac{1}{2}\sum_{j,k} c^i_{jk}\, \omega^j \wedge \omega^k = \sum_{j,k} c^i_{jk}\, \omega^k \wedge \omega^j$$

Solution : If $\{X_1, X_2, ..., X_n\}$ is a basis of g and $\{\omega^1, ..., \omega^n\}$ is the dual basis of g^*, then

i) $\omega^i(X_j) = \delta^i_j$

Hence from theorem 1 above

$$d\omega^i(X_j, X_k) = -\frac{1}{2}\omega^i([X_j, X_k])$$

$$= -\frac{1}{2}\omega^i\left\{\sum c^m_{jk} X_m\right\} \text{ from Exercise 2 of Q 2.2}$$

$$= -\frac{1}{2}\Sigma c^m_{jk}\, \omega^i(X_m) = -\frac{1}{2}\Sigma c^m_{jk}\, \partial^i_m$$

$$= -\frac{1}{2}c^i_{jk} \text{ by i)}$$

Again from § 1.11

$$\sum_{m,n} c^i_{mn} (\omega^m \wedge \omega^n)(X_j, X_k) = \frac{1}{2}\sum_{m,n} c^i_{mn} \left\{\omega^m(X_j)\omega^n(X_k) - \omega^m(X_k)\omega^n(X_j)\right\}$$

$$= \frac{1}{2}\sum_{m,n} c^i_{mn} \left\{\delta^m_j \delta^n_k - \delta^m_k \delta^n_j\right\}$$

$$= \frac{1}{2}\left\{c^i_{jk} - c^i_{kj}\right\}$$

$$= \frac{1}{2}\left\{c^i_{jk} + c^i_{jk}\right\} \text{ by i) of Exercise 1 of § 2.2}$$

$$= \frac{1}{2}\cdot 2\, c^i_{jk}$$

$$= c^i_{jk}$$

Thus $d\omega^i(X_j, X_k) = -\frac{1}{2}\sum_{m,n} c^i_{mn}\ \omega^m \wedge \omega^n(X_j, X_k), \quad \forall x_j, x_k$

or $\quad d\omega = -\frac{1}{2}\sum_{m,n} c^i_{mn}\ \ \omega^m \wedge \omega^n$

or $\quad d\omega^i = -\frac{1}{2}\sum_{j,k} c^i_{jk}\ \ \omega^j \wedge \omega^k$

Take $i, j, k = 1, 2, 3$, then

$$\sum_{j,k} c^i_{jk}\ \omega^j \wedge \omega^k = c^i_{12}\ \omega^1 \wedge \omega^2 + c^i_{13}\ \omega^1 \wedge \omega^3 + c^i_{21}\ \omega^2 \wedge \omega^1 + c^i_{23}\ \omega^3 \wedge \omega^2$$

$$+ c^i_{31}\ \omega^3 \wedge \omega^1 + c^i_{32}\ \omega^3 \wedge \omega^2$$

$$= 2\, c^i_{12}\ \omega^1 \wedge \omega^2 + 2\, c^i_{13}\ \omega^1 \wedge \omega^3 + 2\, c^i_{23}\ \omega^2 \wedge \omega^3$$

as $c^i_{jk} = -c^i_{hk}$

$$= 2\sum_{j<k} c^i_{jk}\ \ \omega^j \wedge \omega^k$$

Thus, we write

$$d\omega^i = -\sum_{j<k} c^i_{jk} \ \omega^j \wedge \omega^k$$

Hence

$$d\omega^i = +\sum_{j<k} c^i_{jk} \ \omega^k \wedge \omega^j.$$

§. 2.4 Automorphism :

A mapping, denoted by σ_a for every $a \in G$, $\sigma_a : G \to G$ defined by

$$\sigma_a(x) = axa^{-1}, \quad \forall\, x \in G$$

is said to be an **inner automorphism** if

i) $\sigma_a(xy) = \sigma_a(x)\, \sigma_a(y)$

ii) σ_a is injective

iii) σ_a is surjective

such σ_a is written as *ada*.

Exercise : Show that if G is a Lie group, $h \in G$, then the map

$$I_h : G \to G$$

defined by

$$I_h\,(k) = hkh^{-1}$$

is an automorphism.

An inner automorphism of a Lie group G is defined by

2.16) $(ada)(x) = axa^{-1}$, $\forall x \in G$

Now, $(L_a R_{a^{-1}})x = L_a(R_{a^{-1}}x) = L_a(xa^{-1}) = axa^{-1} = (ada)(x)$

$\therefore \quad L_a R_{a^{-1}} = ada$

Using 2.3) we get

2.17) $ada = L_a R_{a^{-1}} = R_{a^{-1}} L_a$

Note that *ada* is a diffeomorphism.

Theorem 1 : Every inner automorphism of a Lie group G induces an automorphism of the Lie algebra g of G.

Proof : For every $a \in G$ let us denote the inner automorphism on G by

i) $(ada)(x) = axa^{-1}$, $\forall\, x \in G$

Now for every G, $e \in G$ and from § 1.7 such $ada : G \to G$ induces a differential mapping $(ada)_*$,

$$(ada)_* : T_e(G) \to T_{ada(e)}(G) \equiv T_e(G)$$

Such a mapping is a linear mapping and by Theorem 2 of § 2.2, the Lie subalgebra g of a Lie group G is such that

$$g \cong T_e(G)$$

Thus to show every *ada* induces an automorphism of the Lie algebra g of G we are to show

ii) $(ada)_*$ is a mapping from g to g

iii) $(ada)_*$ is a homomorphism i.e.

$(ada)_*(X+Y) = (ada)_* X + (ada)_* Y$

$(ada)_*(bX) = b(ada)_* X$

$(ada)_*[X, Y] = [(ada)_* X + (ada)_* Y]$, $\forall\, X, Y$ in g

iv) $(ada)_*$ is injective

v) $(ada)_*$ is surjective

ii) Let $Y \in G$. Then on using 2.17) we get

$(ada)_* Y = (R_{a^{-1}} \circ L_a)_* Y = (R_{a^{-1}})_* (L_a)_* Y$ as $(f \circ g)_* = f_* \circ g_*$

$= (R_{a^{-1}})_* Y$

Thus

vi) $(ada)_* = (R_{a^{-1}})_*$

Again, $(L_p)_* \{(R_{a^{-1}})_* Y\} = \{(L_p)_* (R_{a^{-1}})_* Y\}$, for every $p \in G$

$$=\left(L_p \circ R_{a^{-1}}\right)_* Y$$

$$=\left(R_{a^{-1}} \circ L_p\right)_* Y \qquad \text{by 2.3)}$$

$$=\left\{\left(R_{a^{-1}}\right)_* \circ \left(L_p\right)_*\right\} Y$$

$$=\left(R_{a^{-1}}\right)_* \left(L_p\right)_* Y$$

$$=\left(R_{a^{-1}}\right)_* Y \quad \text{as } Y \in g$$

Consequently, from above, it follows that $\left(R_{a^{-1}}\right)_* Y \in g$.

Hence $(ada)_*$ is a mapping from g to g.

iii) From § 1.7) we know that such $(ada)_*$ is a linear mapping

i.e.

$$(ada)_*(X+Y) = (ada)_* X + (ada)_* Y$$

$$(ada)_*(bX) = b(ada)_* X, \quad b \in R$$

Further, such $(ada)_*$ satisfies

$$(ada)_*[X, Y] = \left[(ada)_* X,\ (ada)_* Y\right]$$

Thus $(ada)_*$ is a homomorphism from g to g.

iv) Clearly $(ada)_*$ is injective, on using vi) and the fact that $R_{a^{-1}}$ is a translation on G.

v) For every $a \in G$, $a^{-1} \in G$ and we set

$(ada^{-1})_* X = Y$, where $X \in G$

we will show that $Y \in G$ and $(ada)_* Y = X$. Now, for $s \in G$,

$$(L_s)_* Y = (L_s)_*(ada^{-1})_* X = (L_s)_*(R_a \circ La^{-1})_* X \qquad \text{by (2.17)}$$

$$=(L_s)_*\{(R_a)_* (L_{a^{-1}})_*\} X$$

$$=(L_s)_* \circ (R_a)_* X$$

$= (L_g \circ R_a)_* X = (R_a \circ L_g)_* X = (R_a)_* X$

$= (ada^{-1})_* X$

$= Y$ as defined.

Thus $Y \in g$

Finally

$(ada)_* Y = \left(L_a \circ R_{a^{-1}}\right)_* Y$ by (2.17)

$= \left(L_a \circ R_{a^{-1}}\right)_* (ada^{-1})_* X$ as defined

$= \left(L_a \circ R_{a^{-1}}\right)_* \left(R_a \circ L_{a^{-1}}\right)_* X$ by (2.17)

$= \left(L_a \circ R_{a^{-1}} \circ R_a L_{a^{-1}}\right)_* X$ by (1.7)

$= (L_e)_* X$ by (2.3) , where $(L_e)_*$ is the identity differential

$= X$

Consequently, $(ada)_*$ is a surjective mapping.

Combining ii) — v), we thus claim

$(ada)_* : g \to g$

is a Lie algebra automorphism.

This completes the proof.

Note : We also write

$(ada)_* = \text{Ad}a$, for every $a \in g$.

and $a \to \text{Ad}a$

is called the **Adjoint representation** of G to g.

§. 2.5 One parameter subgroup of a Lie group

Let a mapping

$a : \text{R} \to \text{G}$

denoted by $a : t \to a(t)$

be a differentiable curve on G. If for all s, t in R

$$a(t)\, a(s) = a(t+s)$$

then the family $\{ a(t) \mid t \in R \}$ is called a one-parameter subgroup of G.

Exercises : 1. Let H $= \{ a(t) \mid t \in R \}$ be a one-parameter subgroup of a Lie group G. Show that H is a commutative subgroup of G.

2. If X is a left invariant vector field on G, prove that, it is complete

We set

2.18) $a(t) = a_t = \phi_t(e)$

where $\{ \phi_t : t \in \mathrm{R} \}$ is one parameter group of transformations on G, generated by the left invariant vector field x.

Exercises : 3. Let $\{ \phi_t \mid t \in \mathrm{R} \}$ be a one-parameter group of transformations on G, generated by $X \in g$ and $\phi_t(e) = a(t)$. If for every $s \in g$,

$$\phi_t \circ L_s = L_s \circ \phi_t$$

show that the set $\{ a(t) \mid t \in R \}$ is a one-parameter subgroup of G and

$\phi_t = R_{a_t}$ holds, for all $t \in R$

4. Let the vector field X be generated by the one parameter group of transformations $\{ R_{a_t} \mid t \in R \}$ on G. Show that X is left invariant on G.

Solution : As $\{ \phi_t \mid t \in \mathrm{R} \}$ is a one-parameter group of transformations on G and $a : t \in R \to a(t) \in \mathrm{G}$ is a differentiable mapping, by definition

$a(t) \cdot a(s) = L_{a(t)}(a(s))$

$= L_{a(t)}(\phi_s(e))$, as defined in the hypothesis

$= (L_{a(t)} \circ \phi_s)(e)$

$= (\phi_s \circ L_{a(t)})(e)$ by the hypothesis

$$= \phi_s\left(L_{a(t)}{}^{(e)}\right)$$

$$= \phi_s(a(t)e)$$

$$= \phi_s(a(t))$$

$= \phi_s(q_t(e))$ as defined

$$= (\phi_s \circ \phi_t)(e)$$

$= \phi_{s+t}(e)$ is $\{\phi(t)\}$ a one-parameter group of transformations on G

$= \phi_{t+s}(e)$, as $s+t = t+s$ in R

$$= a(t+s)$$

Thus the set $\{a(t) \mid t \in R\}$ is a one-parameter subgroup of G.

Again $\phi_t(s) = \phi_t(se) = \phi_t\left(L_s(e)\right) = (\phi_t \circ L_s)(e) = L_s\left(\phi_t(e)\right) = L_s(a_t)$ by (2.18)

$$= sa_t$$

or $\quad \phi_t(s) = R_{a_t}(s), \quad \forall s \in G$

$$\therefore\ \phi_t = R_{a_t}$$

4. From Exercise 3 above

$$R_{a_t} = \phi_t$$

As it is given that $\left\{R_{a_t} \mid t \in R\right\}$ generates the vector field X, from § 1.9, we can say that X_s is the tangent vector to the curve R_{a_t} and we write

$$X_s f = \lim_{t \to 0} \frac{1}{t}\left\{f\left(R_{a_t}(s)\right) - f(s)\right\}$$

$$= \lim_{t \to 0} \frac{1}{t}\left\{f\left(L_q(R_{a_t}(q^{-1}s))\right) - f(L_q)(q^{-1}s)\right\}$$

$$= \lim_{t \to 0} \frac{1}{t} \left\{ f\left(L_q(R_{a_t}(q^{-1}s))\right) - (f \circ L_q)(q^{-1}s) \right\}$$

$$= \lim_{t \to 0} \frac{1}{t} \left\{ (f \circ L_q)\left(R_{a_t}(q^{-1}s)\right) - (f \circ L_q)(q^{-1}s) \right\}$$

i) $\quad X_s f = X_{q^{-1}s}(f \circ L_q)$ from § 1.9

We are left to prove that $X \in g$. Note that, for $q \in g$.

$$L_q : G \to G$$

is a left translation on G and $(L_q)_* : T_p(G) \to T_{L_q(p)}(G) \equiv T_{qp}(G)$ is its differential. Hence

$$\left((L_q)_* X\right) f = X_p(f \circ L_q) \text{ by § 1.7, where } f \in F(G)$$

or $\quad \left((L_q)_* X\right)_{L_q(p)} f = X_p(f \circ L_q)$

If $L_q(p) = s$, then $p = L_q^{-1}(s) = L_{q^{-1}}(s)$ by (2.5)

$$\therefore \quad p = q^{-1}s$$

Consequently, the above equation reduces to

$$\left((L_q)_* X\right)_s f = X_{q^{-1}s}(f \circ L_q) = X_s f \quad \text{by i)}$$

$$\therefore \quad \left((L_q)_* X\right)_s = X_s, \quad \forall s \in G$$

$\therefore \quad (L_q)_* = X$, which shows that X is left invariant.

Theorem 1 : If X, Y $\in g$, then

$$[Y,X] = \lim_{t \to 0} \frac{1}{t} \left\{ (Ada_t^{-1})Y - Y \right\}$$

Proof : Every X $\in g$ induces $\{\phi_t | t \in R\}$ as its 1-parameter group of transformations on G. Hence by §1.9.

$$[Y,X] = -[X,Y] = \lim_{t \to 0} \frac{1}{t} \left\{ (\phi_t)_* Y - Y \right\}$$

Now from § 2.4

$$\left(Ada_t^{-1}\right)Y = \left(ada_t^{-1}\right)_* Y$$

$$= \left(R_{a_t} \circ L_{a_t^{-1}}\right)_* Y \text{ by } 2.17)$$

$$= \left(R_{a_t}\right)_* \left\{\left(L_{a_t^{-1}}\right)_* Y\right\}$$

$$= \left(R_{a_t}\right)_* Y \quad , \text{as } Y \in g$$

$$= (\phi_t)_* Y \text{ by Exercise 3.}$$

Consequently, the above question reduces to,

$$[Y,X] = \lim_{t \to 0} \frac{1}{t}\left\{\left(Ada_t^{-1}\right)Y - Y\right\}$$

§ 2.6 Lie Transformation group (Action of a Lie group on a Manifold)

A Lie group G is a Lie transformation group on a manifold M or G is said to act differentiably on M if the following conditions are satisfied :

i) Each $a \in G$ induces a transformation on M, denoted by

$$p \to pa, \ \forall p \in M.$$

ii) $(a, p) : G \times M \to pa \in M$ is a differentiable map.

iii) $p(ab) = (pa)b \quad , \ \forall\, a, b \in G, \ p \in M.$

We say that **G acts on M on the right.**

Similarly, the action of G on the left can be defined.

Exercise : 1. Let $G = GL_2(R)$ and $M = R$ and

$$\theta : G \times M \to M$$

be a differentiable mapping defined by

$$\theta\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, p\right) = ap + b, \ \ a > 0, \ \ a, b \in R$$

Show that θ is an action on M.

Solution : In this case, $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in G$ and

i) $\theta\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, p\right) = 1 \cdot p + 0, = p$

ii) $\left(\theta\begin{pmatrix}a' & b'\\ 0 & 1\end{pmatrix}, \left(\theta\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}, p\right)\right) = \left(\theta\begin{pmatrix}a' & b'\\ 0 & 1\end{pmatrix}, ap+b\right)$ as defined

$$= a'(ap+b)+b', \text{ as defined}$$

$$= a'ap + a'b + b',$$

$$= \theta\left(\begin{pmatrix}aa' & a'b+b'\\ 0 & 1\end{pmatrix}, p\right) \text{ as defined}$$

$$= \theta\left(\begin{pmatrix}a' & b'\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}, p\right)$$

Thus θ is an action on M.

Definition : If G acts on M on the right such that

2.19) $pa = p, \ \forall p \in M$ implies that $a = e$

then, G is said to act **effectively** on M.

Note : There is no transformation, other than the identity one, which leaves every point fixed.

If G acts on M on the right such that

2.20) $pa = p, \ \forall p \in M$, implies that $a = e$ for some $p \in M$ then, G is said to act **freely** on M.

Note : In this case, it has isolated fixed points.

Theorem 1 : If G acts on M, then the mapping

$$\sigma : g \to \chi(M)$$

denoted by

$\sigma : A \to \sigma(A) = A^*$

is a Lie Algebra homomorphism

Note : $\sigma(A)$ is called the **fundamental vector field** on M corresponding to $A \in g$.

Proof : For every $p \in G$ let

$$\sigma_p : G \to M$$

be a mapping such that

i) $\sigma_p(a) = pa$

Such a mapping is called the **fundamental map** corresponding to $p \in M$.

We want to show that

$$\sigma : g \to \chi(M)$$

is a Lie Algebra homomorphism i.e. we are to prove

ii) $\sigma(X + Y) = \sigma(X) + \sigma(Y)$

iii) $\sigma(bX) = b\sigma(X),\ b \in R$

iv) $\sigma[X, Y] = [\sigma X, \sigma Y]$

It is evident from i) that

v) $\sigma_p(a) = pa = R_a(p)$

Let $A \in g$. Then from §2.5, A generates $\{\phi_t \mid t \in R\}$ as its 1-parameter group of transformation on G, such that

$$a(t) \equiv a_t = \phi_t(e)$$

In this case, such $a(t)$ is the integral curve of A on G. The map

$$(\sigma_p)_* : T_e(G) \to T_{\sigma_p(e)}(M) \equiv T_p(M)$$

is the differential map of σ_p and is a linear mapping by definition such that $(\sigma_p)_* X_e \in T_p(M)$.

Using the hypothesis of the theorem

vi) $(\sigma_p)_* A_e = \{\sigma(A)\}_{\sigma_p(e)} = \{\sigma(A)\}_p = A^*_p$

Note that for every A, B, in g, A + B is in g and hence

$$\{\sigma(A+B)\}_p = (\sigma_p)_*(A+B)_e = (\sigma_p)_*(A_e + B_e) = (\sigma_p)_* A_e + (\sigma_p)_* B_e, \text{ as } (\sigma_p)_* \text{ is linear}$$

$$= \{\sigma(A)\}_p + \{\sigma(B)\}_p$$

$\therefore \quad \sigma(A+B) = \sigma(A) + \sigma(B), \quad \forall\, p \in M.$

Also for $b \in R$ $bA \in g$ and hence

$$\{\sigma(bA)\}_p = (\sigma_p)_*(bA)_e = (\sigma_p)_*(A)_e = b(\sigma_p)_* A_e = b\{\sigma(A)\}_p$$

$\therefore \quad \sigma(bA) = b\sigma(A)$

Thus σ is a linear mapping

Now A_e is the tangent vector to the curve $a(t) \equiv a_t$ at $a(0) = e$. Consequently by §1.7, the vector field $(\sigma_p)_* A_e \in T_{\sigma_p(e)}(M) \equiv T_p(M)$ is defined to be the tangent vector to the curve $\sigma_p(a_t) = pa_t = R_{a_t}(p)$ at $\sigma_p(a_o) = \sigma_p(e) = p$. consequently, by vi), we see that A_e^* induce $R_{a_t} p$ as its one-parameter group of transformations on M.

Again $[\sigma(A), \sigma(B)]_p = [A^*, B^*]_p$

$$= \lim_{t \to 0} \frac{1}{t}\left\{ B_p^* - \left((R_{a_t})_* B^*\right)_p \right\} \text{ by Theorem 3 of §1.9}$$

$$= \lim_{t \to 0} \frac{1}{t}\left\{ (\sigma_p)_* B_e - (R_{a_t})_* B_q^* \right\} \text{ say, where}$$

vii) $\quad p = R_{a_t}(q)$

viii) $\quad$ or $q = (R_{a_t})^{-1} p = R_{a_t^{-1}}(p) = pa_t^{-1}$

Thus $(R_{a_t})_* B_q^* = (R_{a_t})_* B^*_{pa_t^{-1}}$ by vii) above

$$= (R_{a_t})_* (\sigma_{pa_t^{-1}})_* B_e \text{ by vi)}$$

$$= (R_{a_t} \circ \sigma_{pa_t^{-1}})_* B_e \text{ where } R_{a_t} \circ \sigma_{pa_t^{-1}} : G \to M$$

Hence for $b \in G$

$$\left(R_{a_t} \circ \sigma_{pa_t^{-1}}\right)(b) = R_{a_t}\left(\sigma_{pa_t^{-1}}(b)\right)$$

$$= R_{a_t}\ (pa_t^{-1}b) \text{ by i)}$$

$$= pa_t^{-1}ba_t \text{ by definition}$$

$$= \sigma_p\left(a_t^{-1}\, ba_t\right) \text{ by i)}$$

$$= \sigma_p\left(ada_t^{-1}(b)\right) \text{ by 2.16) of § 2.4}$$

$$= \left(\sigma_p \circ ada_t^{-1}\right)(b)$$

$$\therefore \quad R_{a_t} \circ \sigma_{pa_t^{-1}} = \sigma_p \circ ada_t^{-1}$$

Consequently, $\left(R_{a_t}\right)_* B_q^* = \left(R_{a_t} \circ \sigma_{pa_t^{-1}}\right)_* B_e$ reduces to

$\left(R_{a_t}\right)_* B_q^* = \left(\sigma_p \circ ada_t^{-1}\right)_* B_e = \left(\sigma_p\right)_*\left(\left(ada_t^{-1}\right)_* B_e\right) = \left(\sigma_p\right)_*\left(\left(Ada_t^{-1}\right)_* B_e\right)$ from the Note of §2.4

Thus we find

$$\left[\sigma(A), \sigma(B)\right]_p = \lim_{t \to 0} \frac{1}{t}\left\{(\sigma_p)_* B_e - \left(\sigma_p\right)_*\left(\left(Ada_t^{-1}\right)_* B_e\right)\right\}$$

$$= \left(\sigma_p\right)_* \lim_{t \to 0} \frac{1}{t}\left\{B_e - \left(Ada_t^{-1}\right)_* B_e\right\} \text{ as } \left(\sigma_p\right)_* \text{ is a linear mapping.}$$

$$= \left(\sigma_p\right)_* [A, B]_e \text{ by § 1.9}$$

$$= \left(\sigma[A, B]\right)_p \text{ by vi)}$$

$$\therefore \quad \sigma[A, B] = \left[\sigma(A), \sigma(B)\right]$$

Thus the mapping

$$\sigma : g \to \chi(M)$$

is a Lie Algebra homomorphism.

Theorem 2 : If G acts effectively on M, then the map

$$\sigma : g \to \chi(M)$$

defined by

$$\sigma : A \to \sigma(A) = A^*$$

is an isomorphism.

Proof : From Theorem 1, we know that such map $\sigma : g \to \chi(M)$ is a Lie Algebra homomorphism. Hence we are left to prove that

i) σ is injective and ii) σ is surjective.

i) Let A, B $\in g$ and $\sigma(A) = \sigma(B)$ Then

$\sigma(A - B) = \theta$, as σ is a linear mapping.

or $(A - B)^* = \theta$

i.e. $(A - B)^*$ is the null vector on M. Now A–B $\in g$ and it will generate $\{\psi_t(e) \mid t \in R\}$ as its 1-parameter group of transformations on G such that $(A - B)_e$ is the tangent vector to the curve, say

$$b(t) = b_t = \psi_t(e) \text{ at } b(o) = e$$

Consequently, the vector field $(A - B)^* = (\sigma_p)_*(A - B)_e$ is the tangent vector to the curve

$$\sigma_p(b(t)) = pb_t = R_{b_t}(p) \text{ at } \sigma_p(b(o)) = \sigma_p(e) = pe = p.$$

Thus $(A - B)^* = (\sigma_p)_*(A - B)_e$ generates $\{R_{b_t}(p) \mid t \in R\}$ as its 1-parameter group of transformations on M. But $(A - B)^*$ is the null vector on M. Hence the integral curve of $(A - B)^*$ will reduce to a single point of itself. Thus

$$R_{b_t}(p) = p$$

or $pb_t = p$

As G acts effectively on M, comparing this with 2.19) we get, $b_t = e, \ \forall\, p \in M$.

Again $(L_g)_*(A - B) = A - B$ as $(A - B) \in g$

$\therefore\ L_q \circ \psi_t = \psi_t \circ L_q$ from § 1.9

Thus $\psi_t(q) = \psi_t(qe) = \psi_t(L_q(e)) = (\psi \circ L_q)(e) = (L_q \circ \psi_t)(e) = L_q(b_t)$

$$= qb_t = q_e = q$$

Hence from § 1.9

$$(A-B)_q f = \lim_{t \to 0} \frac{1}{t}\{f(\psi_t(q)) - f(q)\} \text{ reduces to}$$

$$(A-B)_q f = \lim_{t \to 0} \frac{1}{t}\{f(q) - f(q)\} = 0.$$

Thus $\quad A - B = 0$

i.e. $A = B$.

Hence $\sigma(A) = \sigma(A)$ implies that $A = B$. Consequently σ is injective.

ii) As G acts effectively on M, σ is surjective.

Thus the map is a Lie Algebra isomorphism and this completes the proof.

Theorem 3 : If G acts freely on M, then, for every non-zero vector field $A \in g$, the vector field A^* on M can never vanish.

Proof : If possible, let A^* be a null vector on M. Then, as done in the previous theorem, every $A \in g$ will generate $\{\psi_t(e) \mid t \in R\}$ as its 1-parameter group of transformations on G and we will have

$$\psi_t(q) = q$$

Consequently from the definition, as given in § 1.9

$$A_q f = \left[\frac{d}{dt} f(\phi_t(q))\right]_{t=0}$$

$$= \lim_{t \to 0} \frac{f(\psi_t(q)) - f(q)}{t}$$

$$= 0.$$

Hence A becomes a null vector, contradicting the hypothesis. Thus the vector field A^* on M can never vanish.

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UNIT - 3

3.1 Linear Connection :

The concept of linear (affine) connection was first defined by Levi-Civita for Riemannian manifolds, generalising the notion of parallelism for Eucliden Spaces. This definition is given in the sense of KOSZUL.

A linear connection on a manifold M is a mapping

$$\nabla : \chi(M) \times \chi(M) \to \chi(M)$$

denoted by

$$\nabla : (X, Y) \to \nabla_X Y$$

satisfying the following conditions :

i) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

ii) $\nabla_{(Y+Z)} X = \nabla_Y X + \nabla_Z X$

iii) $\nabla_{fX} Y = f\, \nabla_X Y$

iv) $\nabla_X (f\, Y) = (Xf)Y + f\, \nabla_X Y, \quad \forall\, X, Y, Z \in \chi(M),\ f \in F(M)$

The vector field $\nabla_X Y$ is called the covariant derivative of Y in the direction of X with respect to the connection

If P is a tensor field of type (o, s) we define

v) $\nabla_X P = XP, \quad \text{if } s = o$

vi) $(\nabla_X P)(Y_1, Y_2, \ldots, Y_n) = X\big(P(Y_1, Y_2, \ldots, Y_n)\big) - \sum_{i=1}^{s} P\left(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s\right)$

Exercise 1 : Let $M = R^n$ and X, Y, $\in \chi(M)$ be such that

$$Y = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i} \quad \text{where } \nabla_X Y = \left(Xb^i\right) \frac{\partial}{\partial x^i}$$

Show that ∇ determines a linear connection on M.

Solution : Let $X = a^i \frac{\partial}{\partial x^i}$, $Z = c^i \frac{\partial}{\partial x^i}$ with $a^i, c^i \in F(M)$, $i = 1, \ldots, n$

Then i) $\nabla_X (Y + Z) = \left(X\left(b^i + c^i\right)\right)\frac{\partial}{\partial x^i}$, as defined

$$= \left(Xb^i + Xc^i\right)\frac{\partial}{\partial x^i} = \left(Xb^i\right)\frac{\partial}{\partial x^i} + \left(Xc^i\right)\frac{\partial}{\partial x^i}$$

$$= \nabla_X Y + \nabla_X Z$$

Similarly it can be shown that

$$\nabla_{(Y+Z)} X = \nabla_Y X + \nabla_Z X$$

Again, $\nabla_{fX}^{Y} = \left(\left(fX\right)b^i\right)\frac{\partial}{\partial x^i} = \left(f\left(Xb^i\right)\right)\frac{\partial}{\partial x^i}$ as $(fY)h = f(Yh)$

$= f\,\nabla_X Y$ and

$\nabla_X (fY) = \left(X\left(f b^i\right)\right)\frac{\partial}{\partial x^i}$ as $= \left((Xf)b^i + f\,(Xb^i)\right)\frac{\partial}{\partial x^i}$ as $X(fg) = (Xf)g + f\,(Xg)$

$$= (Xf)b^i \frac{\partial}{\partial x^i} + f\,(Xb^i)\frac{\partial}{\partial x^i}$$

$$= (Xf)Y + f\,\nabla_X Y$$

Thus ∇ determines a linear connection on M.

Let $(x^1, x^2, \ldots, x^n)$ be a system of co-ordinates in a neighbourhood U of p of M. We define

3.1) $$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^{k} \frac{\partial}{\partial x^k} \text{ where } \Gamma_{ij}^{k} \in F(M)$$

Such Γ_{ij}^{k} are called the christoffel symbols or the connection co-efficients or the components of the connection.

Hence if

$$X = \xi^i \frac{\partial}{\partial x^i}\ ,\ Y = \eta^j \frac{\partial}{\partial x^j}$$ where each $\xi^i, \eta^j \in F(M),\ i = 1, \dots, n$ we see that

$$\nabla_X Y = \nabla_{\xi^i \frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right)$$

$$= \xi^i \nabla_{\frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right) \text{ by iii)}$$

$$= \xi^i\left(\frac{\partial \eta^j}{\partial x^i}\cdot\frac{\partial}{\partial x^j} + \eta^j \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right) \text{ by iv) and 3.1)}$$

3.2) $$\nabla_X Y = \left(\xi^i \frac{\partial \eta^k}{\partial x^i} + \xi^i\, \eta^j \Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}$$

Exercise 2 : Let Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ be the connection co-efficients of the linear connection ∇ with respect to the local coordinate system $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ respectively. Show that in the intersection of the two coordinate neighbourhoods

$$\bar{\Gamma}_{ij}^k = \frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial y^k}{\partial x^l} + \Gamma_{rs}^l \frac{\partial x^r}{\partial y^i}\cdot\frac{\partial x^r}{\partial y^j}\cdot\frac{\partial y^k}{\partial x^l}$$

Solution : In the intersection of the two coordinates

$$\frac{\partial}{\partial y^j} = \frac{\partial x^l}{\partial y^j}\cdot\frac{\partial}{\partial x^l}$$

or $$\frac{\partial y^j}{\partial x^s}\cdot\frac{\partial}{\partial y^j} = \frac{\partial y^j}{\partial x^s}\cdot\frac{\partial x^l}{\partial y^j}\cdot\frac{\partial}{\partial x^l} = \frac{\partial}{\partial x^s}$$

Again, from 3.1) we see that

$$\bar{\Gamma}_{ij}^k \frac{\partial}{\partial y^k} = \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \nabla_{\frac{\partial}{\partial y^i}}\left(\frac{\partial x^l}{\partial y^j}\cdot\frac{\partial}{\partial x^l}\right) \text{ from above}$$

$$=\frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial}{\partial x^l}+\frac{\partial x^l}{\partial y^j}\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial x^l} \quad \text{by iv)}$$

$$=\frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial}{\partial x^l}+\frac{\partial x^l}{\partial y^j}\nabla_{\frac{\partial x^s}{\partial y^i}\cdot\frac{\partial}{\partial x^s}}\frac{\partial}{\partial x^l} \quad \text{from above}$$

$$=\frac{\partial^2 x^l}{\partial y^i \partial y^i}\cdot\frac{\partial}{\partial x^l}+\frac{\partial x^l}{\partial y^j}\cdot\frac{\partial x^s}{\partial y^i}\nabla_{\frac{\partial}{\partial x^s}}\frac{\partial}{\partial x^l} \quad \text{by iii)}$$

$$=\frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial}{\partial x^l}+\frac{\partial x^l}{\partial y^j}\cdot\frac{\partial x^s}{\partial y^i}\Gamma^k_{sl}\frac{\partial}{\partial x^k} \quad \text{by 3.1)}$$

$$=\frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial y^k}{\partial x^l}\cdot\frac{\partial}{\partial y^k}+\frac{\partial x^r}{\partial y^i}\cdot\frac{\partial x^s}{\partial y^i}\Gamma^t_{rs}\frac{\partial}{\partial x^t}$$

Changing $s \to r$

$l \to s$

$k \to t$

$$=\frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial y^k}{\partial x^l}\cdot\frac{\partial}{\partial y^k}+\Gamma^t_{rs}\frac{\partial x^r}{\partial y^i}\cdot\frac{\partial x^s}{\partial y^j}\cdot\frac{\partial y^k}{\partial x^t}\cdot\frac{\partial}{\partial y^k} \quad \text{from above}$$

$$=\left(\frac{\partial^2 x^l}{\partial y^i \partial y^j}\cdot\frac{\partial y^k}{\partial x^l}\cdot+\Gamma^t_{rs}\frac{\partial x^r}{\partial y^i}\cdot\frac{\partial x^s}{\partial y^j}\cdot\frac{\partial y^k}{\partial x^t}\cdot\frac{\partial}{\partial y^k}\right)\frac{\partial}{\partial y^k}$$

Since $\left\{\frac{\partial}{\partial y^k} : k = 1\cdots n\right\}$ is a basis of the tangent space and hence linearly independent and the result follows immediately.

3.2 Torsion tensor field and curvature tensor field on a linear connection

we define a mapping

$T : \chi(M)\times\chi(M)\to\chi(M)$ by

3.2) $T(X,Y)=\nabla_X^Y-\nabla_Y^X-[X,Y]$

and another

$R : \chi(M)\times\chi(M)\times\chi\to\chi(M)$

3.3) $R(X, Y)Z = \nabla_X \nabla_Y^Z - \nabla_Y \nabla_X^Z - \nabla_{[X,Y]}^Z$

Then T is a tensor field of type (1,2) and is called the torsion tensor field and R is a tensor field of type (1, 3), called the curvature tensor field of M.

A linear connection is said to be symmetric if

3.4) $T(X, Y) = 0$

In such case

3.5) $[X, Y] = \nabla_X^Y - \nabla_Y^X$

Exercise : 1. Verify that

i) $T(X, Y) = -T(Y, X)$;

ii) $T(fX + gY, Z) = fT(X, Z) + gT(Y, Z)$;

iii) $T(fX, gY) = fg\, T(X, Y)$.

2. If $\overline{\nabla}_X^Y = \nabla_X^Y - T(X, Y)$, show that $\overline{\nabla}$ is a linear connection and $\overline{T} = -T$

3. Show that

i) $T(T(X, Y), Z) = T(\nabla_X^Y, Z) + T(Z, \nabla_Y^X) - T([X, Y], Z)$

ii) $R(X, X)Y = 0; R(X, Y)Z = -R(Y, X)Z; \quad R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

iii) $R(T(X, Y), Z) = R(\nabla_X^Y, Z) + R(Z, \nabla_Y^X) - R([X, Y], Z)$

iv) $R(X, fY)Z = R(fY, Y)Z = R(X, Y)fZ = f\, R(X, Y)Z$

Hence Show that

$$R(fX, gY)hZ = fgh\, R(X, Y)Z$$

4. **Exercise 3 :** Prove Ricci Identity

a) for a 1-form w :

$$\left(\nabla_X \nabla_Y^{\omega} - \nabla_Y \nabla_X^{\omega} - \nabla_{[X,Y]}^{\omega}\right)Z = -W(R(X, Y)Z)$$

b) for a 2-form W :

$$\left(\nabla_X\nabla_Y^W - \nabla_Y\nabla_X^W - \nabla_{[X,Y]}^W\right)(Z,P) = -W\left(R(X,Y)Z,P\right) - W\left(Z,R(X,Y)P\right)$$

5. If $\left(x^1,\cdots,x^n\right)$ is a local coordinate system and

$$T\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) = T_{ij}^k\frac{\partial}{\partial x^k},\ R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial x^k} = R_{ijk}^h\frac{\partial}{\partial x^h}$$

Show that

i) $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ and $\Gamma_{ij}^k = \Gamma_{ij}^k$ for a symmetric linear connection

ii) $R_{ijm}^k = \frac{\partial}{\partial x^i}\Gamma_{jm}^k - \frac{\partial}{\partial x^j}\Gamma_{im}^k + \Gamma_{jm}^t\Gamma_{ti}^k - \Gamma_{im}^t\Gamma_{jt}^k$

Solution : 1 i) From the definition

$$T(Y,X) = \nabla_Y X - \nabla_X Y - [Y,X]$$

$$= \nabla_Y X - \nabla_X Y + [X,Y]$$

$$= -(\nabla_X Y - \nabla_Y X - [X,Y])$$

$$= -T(X,Y)$$

Thus T is skew-symmetric

ii) $T(fX+gY,Z) = \nabla_{fX+gY}Z - \nabla_Z(fX+gY) - [fX+gY,Z]$

$$= f\,\nabla_X Z + g\nabla_Y Z - \nabla_Z(fX) - \nabla_Z(gY) - [fX,Z] - [gY,Z]$$

$$= g[Y,Z] + (Zg)Y$$

$$= f\{\nabla_X Z - \nabla_Z X - [X,Z]\} + g\{\nabla_Y Z - \nabla_Z Y - [Y,Z]\}$$

$$= fT(X,Z) + gT(Y,Z)$$

Again, using the definition, given in § 3.1 and also from § 1.5 we get

Thus T is a bilinear mapping.

2. To prove that $\overline{\nabla}$ is a linear connection, we have to prove i), ii), iii), iv) of § 3.1. Now

$$\overline{\nabla}_X(Y+Z) = \nabla_X(Y+Z) - T(X,Y+Z) \text{ as defined}$$

$$= \nabla_X Y + \nabla_X Z - T(X,Y) - T(X,Z)$$

$$= \overline{\nabla}_X Y + \overline{\nabla}_X Z, \text{ as defined}$$

similarly, other results can be proved and hence $\overline{\nabla}$ is a linear connection. Now,

$$\overline{T}(X,Y) = \overline{\nabla}_X Y + \overline{\nabla}_Y X - [X,Y], \text{ by definition}$$

$$= \nabla_X Y - T(X,Y) - \nabla_Y X + T(Y,X) - [X,Y], \text{ as defined}$$

$$= T(X,Y) - T(X,Y) - T(X,Y) \text{ by Ex 1 (i) above}$$

$$= -T(X,Y)$$

$$\therefore \overline{T} = -T$$

3. (iv) From the definition

$$R(X,fY)Z = \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z$$

$$= \nabla_X (f\nabla_Y Z) - f\nabla_Y \nabla_X Z - \nabla_{f[XY]+(XfY)} Z$$

$$= (Xf)\nabla_Y Z + f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z - (Xf)\nabla_Y Z$$

$$= f\left(\nabla_X \nabla_Y - \nabla_Y Z \nabla_X Z - \nabla_{[X,Y]} Z\right)$$

$$= fR(X,Y)Z \text{ by definition.}$$

5. From the given condition

$$T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j}\right) - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$$

Using 3.1) we find

$$= \Gamma_{ij}^k \frac{\partial}{\partial x^k} - \Gamma_{ji}^k \frac{\partial}{\partial x^k} - 0$$

or, $T_{ji}^k \frac{\partial}{\partial x^k} = \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) \frac{\partial}{\partial x^k}$, as defined

Since $\left\{\frac{\partial}{\partial x^k} : k = 1, \cdots\cdots, n\right\}$ is a basis and hence linearly independent and thus

i) $\Gamma_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$

If the linear connection is symmetric, then T = 0. consequently, the above equation reduces to

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

ii) From the definition, we see that

$$R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m}=\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^m}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^m}-\nabla_{\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]}\nabla\frac{\partial}{\partial x^m}$$

$$=\nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^k_{jm}\frac{\partial}{\partial x^k}\right)-\nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^k_{im}\frac{\partial}{\partial x^k}\right)\text{ as }\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]=0$$

$$=\left(\frac{\partial}{\partial x^i}\Gamma^k_{jm}\right)\frac{\partial}{\partial x^k}+\Gamma^k_{jm}\Gamma^t_{ik}\frac{\partial}{\partial x^t}-\left(\frac{\partial}{\partial x^j}\Gamma^k_{im}\right)\frac{\partial}{\partial x^k}+\Gamma^k_{im}\frac{\partial}{\partial x^t}$$

Changing the dummy indices $t\to k, k\to t$ in the 2nd and 4th term we get

$$R^k_{ijm}\frac{\partial}{\partial x^k}=\left(\frac{\partial}{\partial x^i}\Gamma^k_{jm}\right)\frac{\partial}{\partial x^k}+\Gamma^t_{jm}\Gamma^k_{it}\frac{\partial}{\partial x^k}-\frac{\partial}{\partial x^j}\Gamma^k_{jm}\frac{\partial}{\partial x^k}-\Gamma^t_{im}\Gamma^k_{jt}\frac{\partial}{\partial x^k}$$

Since $\left\{\frac{\partial}{\partial x^k}:k=1,\cdots\cdots,n\right\}$ is a basis and hence linearly independent, we get from above

$$R^k_{ijm}=\frac{\partial}{\partial x^i}\Gamma^k_{jm}-\frac{\partial}{\partial x^j}\Gamma^k_{im}+\Gamma^t_{jm}\Gamma^k_{it}-\Gamma^t_{im}\Gamma^k_{tj}$$

§ 3.2 Covariant Differential of A Tensor Field of type (o, s)

The covariant differential of a tensor field of type (0, s) is a tensor field of type (0, s + 1) and is defined as

3.6) $(\nabla P)(X_1,X_2,\cdots\cdots,X_{X+1})=\left(\nabla_{X_{x+1}}P\right)(X_1,X_2,;\cdots\cdots,X_S)$

Exercise : 1 Let $\mathfrak{I}^i$ be the components of a vector field Y with respect to a local coordinate system $(x^1,\cdots\cdots,x^n)$ i.e. $Y=\mathfrak{I}^i\frac{\partial}{\partial x^i}$

If $\mathfrak{I}^i;_j$ be the components of the convariant differential ∇Y, so that $\nabla_{\frac{\partial}{\partial x^i}}Y=\mathfrak{I}^i,_j\frac{\partial}{\partial x^i}$

then, show that

$$\mathfrak{I}^i,_j=\frac{\partial\mathfrak{I}^i}{\partial x^j}+\Gamma^i_{kj}\mathfrak{I}^k$$

2. Let ω be a 1 form and $\omega_l d_x l$

   If we write

$$\nabla_{\frac{\partial}{\partial x^i}}\omega\left(\frac{\partial}{\partial x^k}\right)=\omega_{k,i}$$

show that

$$\omega_{k,i} = \frac{\partial \omega_k}{\partial x^i} - \omega_h \Gamma^h_{ki}$$

3. If we write $\left(\nabla_{\frac{\partial}{\partial x^k}} R\right)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} = R^h_{ijm,k}\frac{\partial}{\partial x^h}$

show that $R^h_{ijm,k} = \frac{\partial}{\partial x^k} R^h_{ijm} + R^s_{ijm}\Gamma^h_{sk} - R^h_{sjm}\Gamma^s_{ik} - R^h_{ism}\Gamma^s_{jk} - R^h_{ijs}\Gamma^s_{mk}$

Solution : 1. We write

$$\nabla_{\frac{\partial}{\partial x^j}} Y = \nabla_{\frac{\partial}{\partial x^j}}\left(\eta^i \frac{\partial}{\partial x^i}\right) \quad \text{or} \quad \eta^i_{.j}\frac{\partial}{\partial x^i} = \frac{\partial \eta^i}{\partial x^j}\cdot\frac{\partial}{\partial x^i} + \eta^i \Gamma^k_{ji}\frac{\partial}{\partial x^i}$$

Changing the dummy indices $i \to k$, $k \to i$ in the 2nd term on the r. h. s we get

$$\eta^i_{.j}\frac{\partial}{\partial x^i} = \frac{\partial \eta^i}{\partial x^j}\cdot\frac{\partial}{\partial x^i} + \eta^k \Gamma^i_{jk}\frac{\partial}{\partial x^i}$$

Since $\left\{\frac{\partial}{\partial x^i} \;\therefore\; i = l,....,m\right\}$ is a basis and hence linearly independent and thus we must

have, $\eta^i_{,j} = \frac{\partial \eta^i}{\partial x^j} + \eta^k \Gamma^i_{jk}$

2. As ω is a tensor field of type (0. 1) we have from vi) of ξ.3.1

$$\left(\nabla_{\frac{\partial}{\partial x^i}}\omega\right)\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^i}\left(\omega\left(\frac{\partial}{\partial x^k}\right)\right) - \omega\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right)$$

$$= \frac{\partial}{\partial x^i}\left(\omega_l d_x l\left(\frac{\partial}{\partial x^k}\right)\right) - \omega\left(\Gamma^h_{ik}\frac{\partial}{\partial x^h}\right)$$

$$= \frac{\partial}{\partial x^i}\left(\omega_l \delta^l_k\right) - \Gamma^h_{ik}\,\omega\left(\frac{\partial}{\partial x^h}\right)$$

$$\text{or, } \omega_{k,i} = \frac{\partial}{\partial x^i}\left(\omega_k\right) - \Gamma^h_{ik}\,\omega_h$$

$$\text{Thus, } \omega_{k,i} = \frac{\partial \omega_k}{\partial x^i} - \omega_h \Gamma^h_{ik}$$

3. From the definition

$$\left(\nabla_{\frac{\partial}{\partial x^k}} R\right)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} = \nabla_{\frac{\partial}{\partial x^k}} R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} - R\left(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m}$$

$$- R\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} - R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^m}$$

$$= \frac{\partial}{\partial x^k}\left(R^h_{ijm}\frac{\partial}{\partial x^h}\right) - R\left(\Gamma^s_{ki}\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} - R\left(\frac{\partial}{\partial x^i}, \Gamma^s_{kj}\frac{\partial}{\partial x^s}\right)\frac{\partial}{\partial x^m}$$

$$- R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\Gamma^s_{km}\frac{\partial}{\partial x^s}$$

$$= \left(\frac{\partial}{\partial x^k} R^h_{ijm}\right)\frac{\partial}{\partial x^h} + R^h_{ijm}\Gamma^s_{kh}\frac{\partial}{\partial x^s} - \Gamma^s_{ki}R^h_{sjm}\frac{\partial}{\partial x^h} - \Gamma^s_{kj}R^h_{ism}\frac{\partial}{\partial x^h} - \Gamma^s_{km}R^h_{ijs}\frac{\partial}{\partial x^h}$$

or $R^h_{ijm,k}\frac{\partial}{\partial x^h} = \left(\frac{\partial}{\partial x^k} R^h_{ijm}\right)\frac{\partial}{\partial x^h} + R^s_{ijm}\Gamma^h_{ks}\frac{\partial}{\partial x^h}$

$- \Gamma^s_{ki}R^h_{sjm}\frac{\partial}{\partial x^h} - \Gamma^s_{kj}R^h_{ism}\frac{\partial}{\partial x^h} - \Gamma^s_{km}R^h_{ijs}\frac{\partial}{\partial x^h}$, on changing the dummy indices $h \rightarrow s$, $s \rightarrow h$ in the 2nd term on the right hand side.

Since $\left\{\frac{\partial}{\partial x^h} \therefore h = 1,\ldots,n\right\}$ is a basis and hence linearly independent and thus we must have,

$$R^h_{ijm,k} = \frac{\partial}{\partial x^k} R^h_{ijm} + R^s_{ijm}\Gamma^h_{sk} - R^h_{sjm}\Gamma^s_{ki} - R^h_{ism}\Gamma^s_{jk} - R^h_{ijs}\Gamma^s_{km}$$

UNIT - 4

§.4.1 Riemannian Metric, Riemannian Connection :

Let us define a covariant tensor field of order 2 on M i.e. $g : \chi(M) \times \chi(M) \to F(M)$

Which satisfies

i) $g(X, X) > 0$: positive definite

ii) $g(X, X) = 0$ if and only if $X = \theta$: non singular

iii) $g(X,Y) = g(Y, X)$: symmetry , $\forall X, Y$ in $\chi(m)$

Such g is called a Riemannian metric on M and the differentiable manifold M together with such g is defined to be a **Riemannian Manifold, denoted symbolically by** (M, g).

Let $\{x^1, ;x^2, ..., x^n\}$ be a co-ordinate system is a neighbourhood U of $p \in M$. We define

iv) $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$

Note If we define $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} = \begin{cases} 1, & i = j \\ o, & i \neq j \end{cases}$

then the matrix of g relative to the basis $\left\{\frac{\partial}{\partial x^i}\right\}$ is given by

$$g = \begin{pmatrix} 10.....0 \\ 01......0 \\ .... \\ 00.......1 \end{pmatrix}$$

A linear connection on a Riemanian manifold (M, g) is said to be a **metric connection** if and only if

4.1) $\nabla g = 0$ i.e. $(\nabla_X g)(Y, Z) = 0, \ \forall X, Y, Z$ in $\chi(M)$

The unique metric connection with vanishing torsion is called the **Riemannian Connection** or the **Levi-Civita Connection**. In this case

4.2) $\nabla_X Y - \nabla_Y X = [X, Y]$

Theorem 1 : Every Riemannian manifold (M, g) admits a unique Riemannian Connection.

Proof : To prove the existence of such a connection, let us define a mapping

$\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M)$

denoted by

$\nabla : (X, Y) \rightarrow \nabla_X Y$

as follows

4.3) $2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) + g(X, [Z, Y]) + g(Y, [Z, X])$

Clearly, $2g(\nabla_X Y + Z), W) - 2g(\nabla_X Y, W), 2g(\nabla_X Z, W)$

$= Xg(Y + Z, W) + (Y + Z)g(W, X) - Wg(X, Y + Z) + g([X, Y + Z], W) + g(X, [W, Y + Z])$

$+ g(Y + Z, [W, X]) - Xg(Z, W) - Yg(W, X) + Wg(X, Z) - g([X, Y), W) - g(X, [W, Y])$

$- g(Y, [W, X]) - Xg(Z, W) - Zg(W, X) + Wg(X, Z) - g([X, Z], W)$

$- g(X, [W, Z]) - g(Z, [W, X])$

$= 0$

$\therefore\ 2g(\nabla_X (Y + Z) - \nabla_X Y - \nabla_X Z, W) = 0,$ as g is linear

Whence

$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

Similarly it can be shown that

$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$

$\nabla_{fX} Y = f\, \nabla_X Y,$

$\nabla_X (fY) = (Xf) Y + f\, \nabla_X Y$

Thus such a mapping determines a linear connection on M. Also, from (4.3) it can be shown that

$2Xg(Y,Z) - 2g(\nabla_X Y, Z) - 2g(Y, \nabla_X Z) = 0$

or, $\nabla_X g(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$ by v) of ξ.3.1

or, $(\nabla_X g)(Y,Z) = 0, \ \forall X, Y, Z$

Thus such a linear connection admits a metric connection. Further, it can be shown that

$\nabla_X Y - \nabla_Y X - [X,Y] = 0$

Hence such a metric connection admits a Riemannian connection

To prove the uniqueness, let $\bar{\nabla}$ be another such connection. Then we must have

$Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$ and $\nabla_X Y - \nabla_Y X - [X,Y] = 0$

$Xg(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = 0$ and $\bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = 0$

Subtracting,

$g(\bar{\nabla}_X Y - \nabla_X Y, Z) + g(Y, \bar{\nabla}_X Z - \nabla_X Z) = 0 \ \forall X, Y, Z$ and $\bar{\nabla}_X Y - \bar{\nabla}_X Y = \nabla_Y X - \nabla_Y X$

where form, we get

$\bar{\nabla}_X Y - \nabla_X Y = 0$

$\therefore \nabla_X Y = \bar{\nabla}_X Y$

Thus uniquences is established. This completes the proof

Exercise : 1 In terms of a local coordinate system $\{x^1, x^2, \cdots, x^n\}$ in a neighbourhood U of p of a Riemannian Manifold (M, g) show that

i) the components Γ^i_{jk} defined in UNIT 3 is symmetric and

ii) the Riemannian metric is covariantly constant.

2. Let ∇ be a metric connection of a Riemannian manifold (M, g) and $\tilde{\nabla}$ be another linear connecting given by

$\tilde{\nabla}_X Y = \nabla_X Y + T(X,Y)$

where T is the torsion tensor of M. Show that the following condition are equivalent

i) $\tilde{\nabla} g = 0$ and ii) $g(T(X,Y),Z) + g(Y, T(X,Y)) = 0$

3. In terms of a local coordinate system $\{x^1, \ldots\ldots, x^n\}$ the components Γ^i_{jk} of the Riemannian connection are given by

$$g_{im}\Gamma^i_{jk} = \frac{1}{2}\left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m}\right)$$

Solution : 1. A Riemannian Manifold (M, g) admits a unique Riemannian Connection i.e.

$$T = 0$$

$$\text{or } \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

In terms of a local coordinate system $\{x^1, \ldots, x^n\}$, we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

using 3.1),

$$\Gamma^k_{ij}\frac{\partial}{\partial x^k} - \Gamma^k_{ji}\frac{\partial}{\partial x^k} - 0 = 0$$

$$\text{or } \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right)\frac{\partial}{\partial x^k} = 0$$

Since $\left\{\frac{\partial}{\partial x^k}\right\}$ is a basis and hence linearly independent and thus

$$\Gamma^k_{ij} = \Gamma^k_{ji} \text{ i.e. symmetric.}$$

By definition, on a Riemannian Manifold (M, g),

$$(\nabla_X g)(Y, Z) = 0, \ \forall\ X, Y, Z \text{ in } (M, g)$$

In terms of a local co-ordinate system $\{x^1, \ldots, x^n\}$, taking $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$, we find

$$\left(\nabla_{\frac{\partial}{\partial x^i}} g\right)\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = 0$$

$$\text{or } \frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) - g\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) - g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right) = 0$$

using 3.1) we get

$$\frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) - g\left(\Gamma^l_{ij}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) - g\left(\frac{\partial}{\partial x^j}, \Gamma^l_{ik}\frac{\partial}{\partial x^l}\right) = 0$$

$$\text{or } \frac{\partial}{\partial x^i} g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} = 0 \text{ by iv)}$$

or, $g_{jk,i} = 0$

i.e. Riemannian metric is covariantly constant.

2. Let us assume that i) be true. Then by definition,

$$Xg(Y, Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) = 0$$

Using the condition,

$$Xg(Y, Z) - g(\nabla_X Y + T(X, Y), Z) - g(Y, \nabla_X Z + T(X, Z)) = 0$$

$$\text{or } (\nabla_X g)(Y, Z) - g(T(X, Y), Z) - g(Y, T(X, Z)) = 0$$

Using 4.1), one gets

$$g(T(X, Y), Z) + g(Y, T(X, Z)) = 0$$

Let now the above result be true. Then using the condition

$$g(\tilde{\nabla}_X Y - \nabla_X Y, Z) + g(Y, \tilde{\nabla}_X Z - \nabla_X Z)$$

$$\text{or, } g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Using 4.1) on the right hand side we get

$$g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = Xg(Y, Z)$$

$$\text{or, } g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = \tilde{\nabla}_X g(Y, Z)$$

$$\text{i.e., } (\tilde{\nabla}_X g)(Y, Z) = 0 \ \forall X, Y, Z$$

$$\text{i.e., } \tilde{\nabla} g = 0$$

3. Using iv) we find $g_{im} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m}\right)$

$$\text{or, } 2g_{im}\Gamma^i_{jk} = 2g\left(\Gamma^i_{jk}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m}\right)$$

Using 3.1) and 4.3) one gets the desired result ofter a few steps

Theorem 2 : If R is the curvature tensor of the Riemannian Manifold (M, g), then

4.4) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$: Bianchi's 1st identity

4.5) $(\nabla_U R)(X, Y)Z + (\nabla_X R)(Y, U)Z + (\nabla_Y R)(U, X)Z = 0$: Bianchi's 2nd identity.

4.6) $g(X,Y)Z,U) = -g(R(X,Y)U,Z)$

4.7) $g(R(X,Y)Z,U) = -g(R(Z,U)X,Y)$

Proof : Using 3.3), 3.5) one gets

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ by Jacobi identity

4.5) is Left to the reader

To prove 4.6), one gets from 4.1)

$(\nabla_X g)(Z,U) = 0, \forall X, Z, U$

$\alpha) Xg(Z,U) = g(\nabla_X Z, U) + g(Z, \nabla_X U)$

or, $\nabla_Y (Xg(Z,U)) = \nabla_Y \{g(\nabla_X Z, U) + g(Z, \nabla_X U)\}$

or, $Y(Xg(Z,U)) = Yg(\nabla_X Z, U) + Yg(Z, \nabla_X U)$

using $\alpha)$ on the right side we get

$Y(Xg(Z,U) = g(\nabla_Y \nabla_X, Z, U) + g(\nabla_X Z, \nabla_Y U) + g(\nabla_Y Z, \nabla_Y U) + g(Z, \nabla_Y \nabla_X U)$

Thus, we find

$X(Yg(Z,U)) - Y(Xg(Z,U)) - [X,Y]g(Z,U)$

$= g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla^Z_{[X,Y]}, U\right) + g\left(Z, \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla^U_{[X,Y]}\right)$

$= g(R(X,Y)Z,U) + g(Z,R(X,Y)U)$

Using the definition of [X, Y] f, on the left hand side, one finds

$g(R(X,Y)Z,U) + g(Z,R(X,Y)U) = 0$

Again, $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

$g(R(X,Y)Z) + g(R(Y,Z)X,U) + g(R(Z,X)Y,U) = 0 \ldots\ldots \beta)$

Similarly, we can write

$g(R(U,Z)X,Y)+g(R(Z,X)U,Y)+g(R(X,U)Z,Y)=0$ ....γ)

$g(R(Y,X)U,Z)+g(R(X,U)Y,Z)+g(R(U,Y)X,Z)=0$ ....δ)

$g(R(Z,U)Y,X)+g(R(U,Y)Z,X)+g(R(Y,Z)U,X)=0$ ....ξ)

Adding α), β), γ), δ), ξ) and using 4.6) we get

$g(R(X,Y)Z,U)+g(R(U,Z)X,Y)+g(R(Y,X)U,Z)+g(R(Z,U)Y,X)=0$

Using Exercise 3(ii) § 3.2 in the second and in the third term of the above equation.

or, $g(R(X,Y)Z,U)-g(R(Z,U)X,Y)-g(R(X,Y)U,Z)+g(R(Z,U)Y,X)=0$

After a few steps one gets

$$2g(R(X,Y)Z,U)=2g(R(Z,U)X,Y)$$

i.e. $g(R(X,Y)Z,U)+g(R(Z,U)X,Y)$

Exercise 4. In terms of a local coordinate system $\{x^1,\ldots\ldots,x^n\}$ in a neighbourhood U of p of (M, g) show that

i) $R^m_{ijk}+R^m_{jki}+R^m_{kij}=0$

ii) $R^h_{ijk,m}+R^h_{jmk,i}+R^h_{mik,j}=0$

iii) $R^h_{ijk}g_{hm}=-R^h_{jim}g_{hk}$

iv) $R^h_{ijk}g_{hm}=-R^h_{kmi}g_{hj}$

Solution : i) From ii) of Exercise 5 in § 3.2 and also using the result

$$\Gamma^m_{jk}=\Gamma^m_{kj}$$

the result follows immediately

ii) Left to the reader

ii) using ii) of Exercise 5 in § 3.2, on finds

$$R^h_{ijk}g_{hm}=\left(\frac{\partial}{\partial x^i}\Gamma^h_{jk}-\frac{\partial}{\partial x^i}\Gamma^h_{ik}+\Gamma^t_{jk}\Gamma^h_{ti}-\Gamma^t_{ik}\Gamma^h_{tj}\right)$$

$$=\frac{\partial}{\partial x^i}\left(\Gamma^h_{jk}g_{hm}\right)-\Gamma^h_{jk}\frac{\partial}{\partial x^i}\left(g_{hm}\right)-\frac{\partial}{\partial x^j}\left(\Gamma^h_{ik}g_{hm}\right)+\Gamma^h_{ik}\frac{\partial}{\partial x^j}g_{hm}$$

$$+\Gamma^t_{jk}\Gamma^h_{ti}\,g_{hm}-\Gamma^t_{ik}\Gamma^h_{tj}\,g_{hm}$$

Using Exercise 3 of § 4.1 we get

$$R^h_{ijk}g_{hm} = \frac{1}{2}\cdot\frac{\partial}{\partial x^i}\left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m}\right) - \Gamma^h_{jk}\frac{\partial g_{mh}}{\partial x^i}$$

$$-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m}\right) + \Gamma^h_{ik}\frac{\partial g_{hm}}{\partial x^j} + \frac{1}{2}\Gamma^t_{jk}\left(\frac{\partial g_{mt}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^t} - \frac{\partial g_{ti}}{\partial x^m}\right)$$

$$-\frac{1}{2}\Gamma^t_{ik}\left(\frac{\partial g_{mt}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^t} - \frac{\partial g_{tj}}{\partial x^m}\right)$$

Similarly, one can write $R^h_{ijm}g_{hk}$

Thus, $R^h_{ijk}g_{hm} + R^h_{ijm}g_{hk}$

$$= -\frac{1}{2}\Gamma^h_{jk}\left(\frac{\partial g_{hm}}{\partial x^i} + \frac{\partial g_{hi}}{\partial x^m} - \frac{\partial g_{mi}}{\partial x^h}\right) + \frac{1}{2}\Gamma^h_{ik}\left(\frac{\partial g_{hm}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^m} - \frac{\partial g_{mj}}{\partial x^h}\right)$$

$$-\frac{1}{2}\Gamma^h_{jm}\left(\frac{\partial g_{kh}}{\partial x^i} + \frac{\partial g_{hi}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^h}\right) + \frac{1}{2}\Gamma^h_{im}\left(\frac{\partial g_{hk}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h}\right)$$

$$= -\Gamma^h_{jk}\Gamma^t_{im}g_{th} + \Gamma^h_{ik}\Gamma^t_{jm}g_{th} - \Gamma^h_{jm}\Gamma^t_{ik}g_{th} + \Gamma^h_{im}\Gamma^t_{jk}g_{th}$$

Thus, $R^h_{ijk}g_{hm} + R^h_{ijm}g_{hk} = 0$ or $R^h_{ijk}g_{hm} = -R^h_{ijm}g_{hk}$

iv) From Exercise iii) above we write

$$R^h_{ijk}g_{hm} - R^h_{kmi}g_{hj} = -\frac{1}{2}\Gamma^h_{jk}\frac{\partial g_{mh}}{\partial x^i} + \frac{1}{2}\Gamma^h_{jk}\frac{\partial g_{mi}}{\partial x^h} - \frac{1}{2}\Gamma^h_{jk}\frac{\partial g_{hi}}{\partial x^m}$$

$$+\frac{1}{2}\Gamma^h_{mi}\frac{\partial g_{jh}}{\partial x^k} - \frac{1}{2}\Gamma^h_{mi}\frac{\partial g_{jk}}{\partial x^h} + \frac{1}{2}\Gamma^h_{mi}\frac{\partial g_{hk}}{\partial x^j}$$

$$= -\frac{1}{2}\Gamma^h_{jk}\left(\frac{\partial g_{mh}}{\partial x^i} + \frac{\partial g_{hi}}{\partial x^m} - \frac{\partial g_{mi}}{\partial x^h}\right) + \frac{1}{2}\Gamma^h_{mi}\left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right)$$

$$= -\frac{1}{2}\Gamma^h_{jk}\Gamma^t_{im}g_{th} + \frac{1}{2}\Gamma^h_{mi}\Gamma^t_{jk}g_{th} = 0 \quad \therefore R^h_{ijk}g_{hm} = R^h_{kmi}g_{hj}$$

Theorem 3 : If $\overline{\nabla}$ and ∇ correspond to the Levi-Civita (Riemannian) Connection and the metric connection with non-vanishing torsion T, then

4.8) $\nabla_X Y - \overline{\nabla}_X Y = \frac{1}{2}\{T(X, Y) + T'(X, Y), T'(Y, X)\}$ where

4.9) $g(T(Z, X), Y) = g(T'(X, Y), Z)$

Proof : From 4.1) we see that

$$(\nabla_X g)(Y, Z) = 0 \text{ and } (\overline{\nabla}_X g)(Y, Z) = 0$$

Thus $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ and

$$Xg(Y, Z) = g(\overline{\nabla}_X Y, Z) + g(Y, \overline{\nabla}_X Z)$$

Subtracting these two, we get

$$a) \quad \begin{cases} g(U(X, Y), Z) + g(Y, U(X, Z)) = 0 \quad \text{where} \\ U(X, Y) = \overline{\nabla}_X Y - \nabla_X Y, \\ U(X, Z) = \overline{\nabla}_X Z - \nabla_X Z \end{cases}$$

Again from 4.2) we get

$$0 = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \text{ and}$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Subtracting and using a) above

$$-T(X, Y) = U(X, Y) - U(Y, X)$$

$$\text{or, } g(T(X, Y), Z) = g(U(Y, X), Z) - g(U(X, Y), Z)$$

Again, on using 4.9), we find

$$\begin{aligned} g(T(X, Y), Z) + g(T'(X, Y), Z) + g(T'(Y, X), Z) &= g(T(X, Y), Z) \\ &+ g(T(Z, X), Y) + g(T(Z, Y), X) \\ &= g(U(Y, X), Z) - g(U(X, Y), Z) + g(U(X, Z), Y) - g(U(Z, X), Y) \\ &+ g(U(Y, Z), X) - g(U(Z, Y), X) \\ &= -2g(U(X, Y), Z) \quad \text{by a)} \\ &= -2g(\overline{\nabla}_X Y - \nabla_X Y, Z) = 2g(\nabla_X Y - \overline{\nabla}_X Y, Z) \end{aligned}$$

$$\therefore \nabla_X Y - \overline{\nabla}_X Y = \frac{1}{2}\{T(X, Y) + T'(X, Y), T'(Y, X)\}$$

3.4.2 Riemann Curvature tensor field :

The Riemann Curvature tensor field of 1st kind of M is a tensor field of degree (0, 4), denoted also by R

$R : \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \to F(M)$

and defined by

4.10) $R(X, Y, Z, W) = g(R(X, Y)Z, W), X, Y, Z, W$ in $\chi(M)$

Exercise : 1 Verify that

i) $R(X, Y, Z, W) = -R(Y, X, Z, W)$

ii) $R(X, Y, Z, W) = -R(X, Y, W, Z)$

iii) $R(X, Y, Z, W) = -R(Z, W, X, Y)$

iv) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$

v) $(\nabla_U R)(X, Y, Z, W) + (\nabla_Z R)(X, Y, W, U) + (\nabla_W R)(X, Y, U, Z) = 0$

2. If R^h_{ijk} and g_{hm} are the components of the curvature tensor and the metric tensor with respect to a local coordinate system $x^1, x^2, \dots\dots, x^n$ then the components R_{ijkm} of the Rieman Curvature tensor are given by

$$R_{ijkm} = R^h_{ijk}\, g_{hm}$$

where $R_{ijkm} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}\right)$

3. A vector field z on (M, g) is called a **gradient vector field** if

4.11) $g(Z, Y) = df(Y) = Yf, \; f \in F(M)$

for every vector field Y and M. Show that for such Z

$g(\nabla_X Z, Y) = g(\nabla_Y Z, X)$ for every vector field X on M.

Solution : From 4.1) we see that

$$(\nabla_X g)(Y, Z) = 0 \text{ for all X, Y, Z in } \chi(M)$$

$$\text{or } Xg(Y, Z) - g(\nabla_X Y, Z) = g(Y, \nabla_X Z)$$

Using 4.11), one finds

$$g(\nabla_X Z, Y) = X(Yf) - g(\nabla_X Y, Z)$$

similarly $\quad g(\nabla_Y Z, X) = Y(Xf) - g(\nabla_Y X, Z)$

$\therefore\; g(\nabla_X Z, Y) - g(\nabla_Y Z, X) = X(Yf) - Y(Xf) + g(\nabla_Y X, Z) - g(\nabla_X Y, Z)$

or, $g(\nabla_X Z, Y) - g(\nabla_Y Z, X) = [X, Y]f - g(\nabla_X Y - \nabla_Y X, Z)$

$= [X, Y]f - g[X, Y], Z)$ by 4.2)

$= [X, Y]f - [X, Y]f$ by 4.11)

$= 0$

Thus

$$g(\nabla_X Z, Y) - g(\nabla_Y Z, X)$$

3.4.3 Einstein Manifold :

Let $\{e_1, e_2, \cdots e_n\}$ be an orthonormal basis of $T_p(M)$ Then the Ricci tensor field, denoted by S, is the covariant tensor field of degree 2 and is defined by

$$S(X_p, Y_p) = \sum_{i=1}^{n} R\left(((e_i)_p, X_P, Y_P, (e_i)_P\right)$$

We write it as

4.12) $$S(X, Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i)$$

Such a tensor field S(X, Y) is also called the **Ricci Curvature** of M.

If there is a constant λ such that

4.13) $$S(X, Y) = \lambda g(X, Y)$$

then M is called on **Einstein Manifold.**

The function r on M, defined by

$$r(p) = \sum_{i=1}^{n} S\left((e_i)_p, (e_i)_p\right)$$

is called the **scalar curvature** of M. We write it as

4.14) $$r = \sum_{i=1}^{n} S(e_i, e_i)$$

Exercise : 1. Show that the Ricci tensor field is symmetric.

At any $p \in M$, we denoted by Π a plane section i.e., a two dimensional subspace of $T_p(M)$. The sectional curvature of Π denoted by $K(\Pi)$ with orthonormal basis X, Y is defined as

4.15) $K(\Pi) = g(R(X, Y) Y, X) = R(X, Y, Y, X)$

If $K(\Pi)$ is constant for all plane section and for all points of p M, Then (M, g) is called a manifold of constant curvature. For such a manifold

4.16) $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}$ where $k(\Pi)$ say

Example : Euclidean space is of Constant Curvature

Exercise : 1, Show that a Riemannian manifold of constant curvature is an Einstein Manifold.

2. If M is a 3-dimensional Einstein Manifold, then, it is a manifold of constant curvature

Solution : Let $\{X_1, X_2, X_3\}$ be an orthonormal basis of $T_p(M)$ Then, the sectional curvature with orthonormal basis X_1, X_2 denoted by $K(\Pi_{12})$ is given by

$$K(\Pi_{12}) = R(X_1, X_2, X_2, X_1)$$

$$= R(X_2, X_1, X_1, X_2)$$

$$= K(\Pi_{21})$$

Thus, $K(\Pi_{ij}) = K(\Pi_{ji}), i \neq j$

Again from 4.12)

$$S(X_1, X_2) = \sum_{i=1}^{3} R(X_i, X_1, X_2, X_i)$$

$$= R(X_1, X_1, X_2, X_1) + R(X_2, X_1, X_2, X_1) + R(X_3, X_1, X_2, X_3)$$

$$= 0 + K(\Pi_{21}) + K(\Pi_{31})$$

$$= K(\Pi_{12}) + K(\Pi_{13})$$

$$S(X_2, X_2) = K(\Pi_{21}) + K(\Pi_{23}) \text{ and}$$

$$S(X_3, X_3) = K(\Pi_{31}) + K(\Pi_{32})$$

As it is a 3-dimensional Einstein manifold, so from 4.13)

$$S(X_1, X_1) = \lambda g(X_1, X_1) = \lambda$$

$$S(X_1, X_2) = \lambda g(X_1, X_2) = 0$$

Thus, $S(X_1, X_1) + S(X_2, X_2) - S(X_3, X_3) = 2K(\Pi_{12})$

$$\text{or, } \lambda = 2K(\Pi_{12})$$

$$\therefore\ K(\Pi_{12}) = \frac{\lambda}{2} = \text{constant.}$$

$$\text{i.e. } K(\Pi_{ij}) = \text{Constant},\ i \neq j$$

Thus every 3 deminsional Einstein manifold is a manifold of constant curvature.

§ 4.4 Semi-symmetric Metric Connection

A linear connection is said to be a semi-symmetric connection if

4.17) $T(X,Y) = w(Y)X - w(X)Y$, for every 1-form w.

A linear connection for which

4.18) $\nabla g = 0$

is called a semi-symmetric metric connection.

Theorem 1 : If ∇ and $\overline{\nabla}$ correspond to semi-symmetric connection and Levi-Civita Connection respectively, then,

$$\nabla_X Y - \overline{\nabla}_X Y = w(Y)X - g(X, Y)p$$

Where p is a vector field given by

$$g(X, p) = w(X)$$

Proof : Since ∇ correspond to a semi-symmetric connection, by 4.17)

$$T(Z, X) = w(X)Z - w(Z)X$$

$$g(T(Z, X), Y) = g(w(X)Z - w(Z)X, Y)$$

$$= w(X)\, g(Z, Y) - w(Z)\, g(X, Y)$$

Using Theorem 3 of § 4.1 on the l. h. s. we get

$$g(T'(X, Y), Z) = w(X)\, g(Y, Z) - g(Z, p)\, g(X, Y)$$

$$= g(w(X)Y, Z) - g(Z, g(X, Y)\, p)$$

$$= g(w(X)Y - g(X, Y)p, Z)$$

Whence $T'(X, Y) = w(X)Y - g(X, Y)p$

using the above result in 4.8) we get

$$\nabla_X Y - \bar{\nabla}_Y Y = \frac{1}{2}\{T(X,Y) + \omega(X)Y - g(X,Y)p + \omega(Y)X - g(Y,X)p\}$$

Again using 4.17), one gets

$$\nabla_X Y - \bar{\nabla}_Y Y = \omega(Y)X - g(X,Y)p$$

Exercise 1. If ∇ and $\bar{\nabla}$ correspond to a semi-symmetric connection and the Levi-Civita connection respectively, then for any 1-form ω

$$(\nabla_X \omega) = (\bar{\nabla}_X \omega)Y - \omega(X)\omega(Y) + \omega(p)g(X,Y), \text{ where}$$

$$g(X,p) = \omega(X)$$

2. Let $\bar{\nabla}$ be the Levi-Civita Connection and ∇ be another linear connection such that

$\nabla_X Y = \bar{\nabla}_X Y - \omega(X)Y$ where is a 1-form.

Show that ∇ is a semi-symmetric connection for which $\nabla_X g = 2\omega(X)g$

Hints : 1. Note that

$$(\nabla_X \omega)Y = X\omega(Y) - \omega(\nabla_X Y)$$

Use Theorem 1 in the second term on the right hand side, one gets the desired result.

2. Note that

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$= \bar{\nabla}_X Y - \omega(X)Y - \bar{\nabla}_Y X + \omega(Y)X - [X,Y]$$

$$= \bar{T}(X,Y) + \omega(Y)X - \omega(X)Y, \text{ on using the hypothesis}$$

$$\omega(Y)X - \omega(X)Y, \text{ as } \bar{T} = 0.$$

Again,

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X, Z)$$

$$= Xg(Y,Z) - g(\bar{\nabla}_X Y - \omega(X)Y, Z) - g(Y, \bar{\nabla}_X Z - \omega(X)Z)$$

$$= (\bar{\nabla}_X g)(Y,Z) + 2\omega(X)g(Y,Z), \text{ on using the hypothesis}$$

$$\therefore \nabla_X g = 2\omega(X)g, \text{ as } \bar{\nabla}g = 0.$$

§ 4.5 Weyl Conformal Curvature tensor :

The Weyl conformal curvature tensor, denoted by C, is defined on an n-dimensional Riemannian manifold (M, g) as follows :

4.19) $$\begin{cases} C(X, Y)Z = R(X, Y)Z + \lambda(Y, Z)X - \lambda(X, Z)Y + g(Y, Z)LX - g(X, Z)LY \\ \text{where } \lambda \text{ is defined by} \\ \lambda(X, Y) = -\dfrac{1}{n-2} S(X, Y) + \dfrac{r}{2(n-1)(n-2)} g(X, Y) \text{ and L is a tensor field of} \\ \text{type (1, 1) given by} \\ g(LX, Y) = \lambda(X, Y), \text{ for every vector field X, Y, Z on M} \end{cases}$$

An n-dimensional (n > 3) Riemannian manifold is said to be conformally flat if

4.20) $C(X, Y)Z = 0$

Goldberg's Result :

Let (M, g) be a Riemannian manifold and A be the field of symmetric endomorphism corresponding to the Ricci tensor *S* i.e.

4.21) $g(AX, Y) = S(X, Y)$ for every vector fields X, Y on M. Then

4.22) $$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\{g(Y, Z)AX - g(X, Z)AY + S(Y, Z)X - S(X, Z)Y\}$$

$$+ \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}$$

Proof : Note that

$$g(g(Y, Z)LX, Y) = g(Y, Z)\, g(LX, Y) = g(Y, Z)\, \lambda(X, Y) \text{ by 4.19)}$$

$$= -\frac{1}{n-2} g(Y, Z) S(X, Y) + \frac{rg(Y, Z)}{2(n-1)(n-2)} g(X, Y) \text{ by 4.19)}$$

$$= -\frac{1}{n-2} g(Y, Z) g(AX, Y) + \frac{rg(Y, Z)}{2(n-1)(n-2)} g(X, Y) \text{ by 4.21)}$$

$$\text{or } g(Y, Z)LX = -\frac{g(Y, Z)}{(n-2)}AX + \frac{rg(Y, Z)}{2(n-1)(n-2)}X$$

Using the above result & 4.19) we find

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2} S(Y, Z)X + \frac{rg(Y, Z)}{2(n-1)(n-2)}X + \frac{1}{n-2} S(X, Z)Y$$

$$- \frac{rg(X, Z)Y}{2(n-1)(n-2)} - \frac{g(Y, Z)AX}{n-2} + \frac{rg(Y, Z)X}{2(n-1)(n-2)} + \frac{g(X, Z)}{n-2}AY - \frac{rg(X, Z)Y}{2(n-1)(n-2)}$$

$$\text{Or, } C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)AX - g(X,Z)AY + S(Y,Z)X - S(X,Z)Y\}$$

$$+\frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}$$

Exercise : 1 If an $n(n>3)$ – dimensional Einstein Manifold is conformally flat than

2. If we write

$$R_{ijkl} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$C_{ijkl} = g\left(C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$R_{ij} = S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

show that

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}\{g_{jk}R_{il} - g_{ik}R_{jl} + R_{jk}g_{il} - R_{ik}g_{jl}\}$$

$$+\frac{r}{(n-1)(n-2)}\{g_{jk}g_{il} - g_{ik}g_{jl}\}$$

Hints : 1 Using 4.13) in 4.14, one gets $r = \lambda n$

Alsing above result, 4.13), one gets from 4.21)

$$Ax = \frac{r}{n}x$$

Using 4.20) in 4.22) and also the result deduced above, one gets the desired result after a few steps.

2. Using goldberg's result, one gets from the hypothesis

$$C_{ijkl} = g\left(C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

the desired result.

4.5 Conformally Symmetric Riemannian Manifold :

A Riemannian manifold (M, g) is said to be conformally symmetric if

4.23) $\nabla C = 0$

Where C is the Weyl Conformal Curvature tensor

Theorem 1 : A conformally symmetric manifold is of constant scalar curvature if

$$(\nabla_Z S)(Y,W) = (\nabla_W S)(Y,Z) \text{ for all } Y, Z, W$$

Proof : From 4.22) we see that

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}\{g(Y,Z)g(AX,W) - g(X,Z)g(AY,W) +$$

$$+S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$

Taking co-variant derivative on both sides and using (4.23), we get

$$\therefore\ (\nabla_U R)(X,Y,Z,W) = \frac{1}{n-2}\{g(Y,Z)(\nabla_U S)g(X,W) - g(X,Z)(\nabla_U S)g(Y,W)$$

$$+(\nabla_U S)(Y,Z)g(X,W) - (\nabla_U S)(X,Z)g(Y,W)\}$$

$$-\frac{\nabla_U r}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$

It is known from Exercise 1(v) of ξ 4.2 that

$$(\nabla_U R)(X,Y,Z,W) + (\nabla_U R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,Z) = 0$$

Using the result deduced above, and also the hypothesis one gets

$$\nabla_U r\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} + \nabla_Z r\{g(Y,W)g(X,U) - g(X,W)g(Y,U)\}$$

$$+\nabla_W r\{g(Y,U)g(X,Z) - g(X,U)g(Y,Z)\} = 0$$

Let $\{e_i : i = 1, \cdots\cdots, n\}$ be an orthonormal basis vectors.

Taking the sum for $1 \le i \le n$ for $X = U = e_i$, we get on using the result

$$\nabla_{ei} r g(e_i, z) = \nabla_z r$$

that

$$g(Y,Z)\nabla_w r - g(Y,W)\nabla_z r + ng(Y,W)\nabla_z r - g(Y,W)\nabla_z r + g(Y,Z)\nabla_w r - ng(Y,Z)\nabla_w r = 0$$

or $g(Y,Z)\nabla_w r - g(Y,W)\nabla_z r = 0$

Finally taking the sum for $1 \le i \le n$ for $Y = Z = e_i$, we get

$$\nabla_w r = 0, \ n > 1.$$

Thus the manifold is of constant curvature.

Definition : A linear transformation A is symmetric or skew symmetric according as

4.24)
$$\begin{cases} g(AX, Y) = g(X, AY) \\ \text{or} \\ g(AX, Y) = -g(X, AY) \end{cases}$$

Exercise : 1. Show that for a symmetric linear transformation A and a skew-symmetric linear transformation $\bar{R}$, the new linear transformation T defined by, $T = A. \bar{R} = \bar{R}. A$ is skew - symmetric.

Theorem 2 : For a conformally flat $n(n > 3)$ - dimensional Riemannian manifold, the curvature tensor R is of the form

$$R(X,Y) = \frac{1}{n-2}(AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y$$

where $X \wedge Y$ denotes the skew - symmetric endomarphism of the tangent space at every point defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$$

Proof : Using the hypothesis, we find that

$$(AX \wedge Y)Z + (X \wedge AY) = g(Y,Z)AX - g(X,Z)AY + S(Y,Z)X - S(X,Z)Y$$

As the manifold is conformally flat, we get on using the above result and the hypothesis,

$$R(X,Y)Z = \frac{1}{n-2}\{(AX \wedge Y)Z + (X \wedge AY)Z\} - \frac{r}{(n-1)(n-2)}\{X \wedge Y)Z\}$$

i.e. $R(X,Y) = \frac{1}{n-2}(AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y$

Theorem 3 : If in a conformally flat manifold, for a symmetric linear transformation A,

$$R(X, Y)A = A. R (X, Y)$$

then

$$\left(A^2 - \frac{rA}{n-1}\right) X \wedge X = 0$$

Proof : Note that

$$R(X, Y) = - R(Y, X)$$

As A is symmetric, so by Exercise 1 of this article A. R(X, Y) = R(X, Y). A is skew - symmetric. Thus R(Z, W)A is a skew symmetric linear transformation and from 4.24) we can write

$$g((R(Z, W)A)X, X) = - g(X, (R (Z, W) A) X)$$

or $$g(R(Z, W)A)X, X) = - g(X, R (Z, W) AX)$$

$$= - g(R(Z, W) AX, X), \text{ as g is symmetric.}$$

$$\therefore g(R(Z, W)AX, X) = 0$$

Using 4.7) one gets

$$g(R (AX, X)Z, W) = 0$$

Whence $$R(AX, X)Z = 0$$

i.e., $$R(AX, X) = 0$$

Again $(AX \wedge AX)Z = 0$ i.e., $AX \wedge AX = 0$ for every Z.

Using Theorem 2, one gets

$$R(X,AX) = \frac{1}{n-2}(AX \wedge AX + X \wedge A^2X) - \frac{r}{(n-1)(n-2)} X \wedge AX$$

AS R(AX, X) = –R(X, AX) and R(AX, X) = 0, we get from above,

$$X \wedge A^2X - \frac{r}{n-1} X \wedge AX = 0$$

Note that $X \wedge Y$ is skew - symmetric and thus

$$A^2X \wedge X - \frac{r}{n-1} AX \wedge X = 0$$

$$\therefore \left(A^2 - \frac{r}{n-1}\right) X \wedge X = 0$$

Definition : A curve $\sigma = x(t), a \le t \le b$ is called a geodesic on M with a linear connection ∇ if

4.25) $\nabla_X X = 0$

Where X is the vector tangent to the integral curve σ at x(t). Note that the integral curves of a left invariant vector fields are geodesic.

4.7 Biinvariant Riemannian metric on a Lie group :

A Riemannian metric g on a Lie group is said to be biinvariant if it is both left and right invariants.

Exercise 1 : If g is a left invariant convariant tensor field of order 2 on G and X, Y are left invariant vector fields on G, show that g(X, Y) is a constant function.

Theoxem 1 : If G is a Lie group admitting a biinvariant Riemannian metric g, then

4.26) g([X, Y], Z) = g(X, [Y, Z])

4.27) $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$

4.28) $g(R(X,Y)Z,W) = -\frac{1}{4} g([X,Y],[Z,W])$

Proof : Since X, Y are left invariant vector fields, X + Y is also so and hence from 4.25)

$$\nabla_{X+Y}^{X+Y} = 0$$

Using 4.25, we find from above

i) $$\nabla_X Y + \nabla_Y X = 0$$

since M admits a unique Riemannian connection, we must have

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

ii) or $\nabla_X Y = \frac{1}{2}[X, Y]$ from i)

Now for a Riemannian Manifold $(\nabla_Y g)(X, Z) = 0$

or, $$Yg(X, Z) - g(\nabla_Y X, Y) - g(X, \nabla_Y Z) = 0$$

Using Exercise 1 of this article and Exercise 2 of ξ 1.4 we see that

$$Y.\, g(X, Z) = 0$$

Thus from ii) we find that $-\frac{1}{2}g([Y, X]Z) - \frac{1}{2}g(X, [Y, Z]) = 0$

or, $$g([X, Y], Z) - g(X, [Y, Z])$$

or, $$g([X, Y], Z) = g(X, [Y, Z])$$

Again from the definition

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla^Z_{[X,Y]}$$

$$= \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z] \text{ by using ii)}$$

$$= \frac{1}{4}[X, [Y, Z]] + \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z]$$

$$= -\frac{1}{4}[Z, [X, Y]] - \frac{1}{2}[[X, Y], Z] \text{ by Jacobi Identity}$$

$$= \frac{1}{4}[[X, Y], Z] - \frac{1}{2}[[X, Y], Z]$$

$$= -\frac{1}{4}[[X, Y], Z]$$

Again $R(X, Y)Z, W) = -\frac{1}{4}g\big([[X, Y], Z],\ W\big)$ by 4.27)

$$= -\frac{1}{4}g\big([X, Y], Z],\ [Z, W]\big) \text{ by 4.26)}$$

This completes the proof.

Theorem 2 : If G is a Lie group admitting a biinvariant Riemannian metric g and Π is a plane section in $T_p(M)$ where Π is determined by orthonormal left invariant vector fields X, Y at p on G, then the sectional curvature at p is zero if and only if [X, Y] = 0.

Proof : From 4.15)

$$K(\Pi) = g(R(X,Y,)Y,X)$$

$$= -\frac{1}{4}g([X,Y],[Y,X]) \text{ by 4.28)}$$

$$= \frac{1}{4}g([X,Y],[X,Y])$$

The result follows immediately as g is nonsingular.

Theorem 3 : If G is a Lie group admitting a biinvariant Riemannian metric g, then for all left invariant vector fields, X, Y, Z, W, P.

Proof : From Jacobi's identity

$$[W, [P, Z]] + [P, [Z, W]] + [Z, [W, P]] = 0$$

Taking P = [X, Y], we get

$[W, [[X, Y], Z] + [[X, Y], [Z, W]] + [Z, [W, [X, Y]]] = 0$

or $[W, [[X, Y], Z]] - [[X, Y], [W, Z]] = [[W, [X, Y]], Z]$

$= [-[X, [Y, W]] - [Y, [W, X]], Z]$ by Jacobi Identity

i) $[W, [[X, Y], Z]] - [[X, Y], [W, Z]] = [[X, [W, Y]], Z] + [[W, X], Y], Z]$

Again from the definition

$$(\nabla_W R)(P,Z,X,Y) = \nabla_W R(P,Z,X,Y) - R(\nabla_W P,Z,X,Y) - R(P,\nabla_W Z,X,Y) -$$

$$-R(P,Z,\nabla_W X,Y) - R(P,Z,X,\nabla_W Y)$$

$$= 0 + R(X,Y,Z,\nabla_W P) + R(X,Y,\nabla_W Z,P) + R(\nabla_W X,Y,Z,P)$$

$$+P(X,\nabla_W Y,Z,P)$$

Using 4.28), one gets

$$(\nabla_W R)(P,Z,X,Y) = -\frac{1}{8}g([X,Y],[Z,[W,P]]) - \frac{1}{8}g([[W,Z],P],[X,Y])$$

$$-\frac{1}{8}g([[W,X]Y][Z,P]) - \frac{1}{8}g([X,[W,Y],[Z,P]])$$

Using 4.26) successively we get

$$= -\frac{1}{8}\{g([[[X,Y],Z],W],P) + g([[X,Y],[W,Z]],P)$$

$$+g([[W,X],Z],P) + g([[X,[W,Y]],P])\}$$

$$= +\frac{1}{8}g([W,[[X,Y],Z]],P) - \frac{1}{8}g([[X,Y][W,Z]],P)$$

$$-\frac{1}{8}g([[X,[W,Y]],Z],P) - \frac{1}{8}g([[[W,X],Y],Z],P)$$

= 0 by i) for all left invariant vector fields X, Z, Y, W, P.

This completes the proof.

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NOTES

NOTES

NOTES

মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

— রবীন্দ্রনাথ ঠাকুর

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নুতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

— সুভাষচন্দ্র বসু

Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

— Subhas Chandra Bose

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PREFACE

In the curricular structure introduced by this University for students of Post- Graduate diploma programme, the opportunity to pursue Post-Graduate Diploma course in any Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate Diploma level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Cooperation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

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Notification

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Open University

PG (MT)—XA(II)

Fluid Mechanics

Unit 1 □ Irrotational Motion of an Ideal Fluid in Two-Dimensions

Structure

1.0 Introduction

In this chapter, we consider the two-dimensional irrotational steady flow of an ideal incompressible fluid. For plane flow, all dynamic computations for the hydrodynamic considerations, we take a layer of unit height cut by two planes parallel to the plane of the flow. In considering the plane problem, we direct our attention on the kinetic flow around a body fixed in a flow or for the motion of a body in a fluid at rest. We shall restrict our discussions on cylindrical bodies having circular and elliptic cross-sections.

1.1 Irrotational Motion in Two Dimensions. The Stream Function

If the motion of a liquid remains the same in all planes parallel to that of xy and there is no velocity parallel to the z-axis, i.e. if the velocity components u, v are functions of x, y only and the component w = 0, then the motion is said to be two-dimensional and in such a case, we consider the circumstances in the xy-plane. When we speak of the flow across a curve in this plane, we mean the flow is across a unit length of a cylinder whose trace on the xy plane is the curve in question, the generators of the cylinder being parallel to the axis of z. Here the differential equation of the lines of flow is

$$v dx - u dy = 0 \qquad (1)$$

while the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \text{ i.e. } \frac{\partial u}{\partial x}+\frac{\partial}{\partial y}=(-v). \qquad (2)$$

This equation shows that the left hand side of (1) is an exact differential $d\psi$, say. Thus

$$v dx - u dy = d\psi = \frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

leading to $$u=-\frac{\partial\psi}{\partial y},\ v=\frac{\partial\psi}{\partial x} \tag{3}$$

This function $\psi(x, y)$ is called the **stream function** or **current function**. It follows that the lines of flow are given by ψ = constant.

Now if the motion of the liquid be irrotational, then there exists a velocity potential $\phi(x, y)$ such that

$$u=-\frac{\partial\phi}{\partial x},\ v=-\frac{\partial\phi}{\partial y} \tag{4}$$

From (3) and (4) we get

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\ \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \tag{5}$$

so that

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}=0$$

which shows that the families of curves ϕ = constant, ψ = constant cut orthogorally at all their points of intersection. These conditions are satisfied if we take $\phi + i\psi$ to be a function of the complex variable $x + iy$.

Now let $\phi + i\psi = f(x + iy)$. Then

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=f'(x+iy),\ \frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=if'(x+iy)=i\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial x}$$

giving $$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\ \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

Thus ϕ and ψ are conjugate functions. If $w = \phi + i\psi = f(z)$, then w is called the **complex potential**.

Noting that

$$\frac{dw}{dz}=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}=-u+iv \tag{6}$$

we have the magnitude of the velocity at any point as $\left|\frac{dw}{dz}\right|$, since

$$\left|\frac{dw}{dz}\right|=\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}^{\frac{1}{2}}=(u^2+v^2)^{\frac{1}{2}}=\text{velocity}. \tag{7}$$

1.2 Boundary Conditions

From (5), it follows that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

where it is assumed the validity of $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$. Thus the stream function ψ must satisfy the Laplace's equation

$$\nabla^2 \psi = 0 \tag{8}$$

at all points of the liquid. This function ψ satisfies the following boundary conditions :

(a) If the liquid is at rest at infinity, we must have $\frac{\partial \psi}{\partial x} = 0$ and $\frac{\partial \psi}{\partial y} = 0$ at infinity.

(b) At any fixed boundary, the normal velocity must be zero, or the boundary must coincide with a stream line ψ = constant.

(c) At the boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to the normal component of the velocity of the cylinder.

We now express the condition (c) by a formula for ψ as follows.

Let a point O of the cross-section of any cylinder be taken as origin. Let U and V be the velocities parallel to the axis of x and y at O and let the cylinder turn with the angular velocity ω. If P(x, y) be any point on the surface of the cylinder, then the velocity components of P are $U - \omega y$ and $V + \omega x$. If θ is the inclination of the tangent at P with Ox, then from the differential calculus, we have

$$\cos\theta = \frac{dx}{ds} \text{ and } \sin\theta = \frac{dy}{ds} \tag{9}$$

Therefore, the outward normal velocity at P

$$= (U - \omega y)\sin\theta - (V + \omega x)\cos\theta$$

$$= (U - \omega y)\frac{dy}{ds} - (V - \omega x)\frac{dx}{ds}. \tag{10}$$

Also the velocity of the liquid in the outward normal is $-\frac{\partial \psi}{\partial s}$.

On equating above two expressions for the normal component of velocities in accordance with condition (c), we have

$$-\frac{\partial \psi}{\partial s} = (U - \omega y)\frac{dy}{ds} - (V - \omega x)\frac{dx}{ds}.$$

Integrating this equation along the arc, we get

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + C \tag{11}$$

where C is an arbitrary constant.

Let the cylinder move along the x-axis with velocity U without rotation (so that V = 0 and $\omega = 0$). Then (11) reduces to

$$\psi = -Uy + C. \tag{12}$$

Similarly, if the cylinder moves along the y-axis with velocity V without rotation, then (11) gives

$$\psi = Vx + C. \tag{13}$$

1.3 Motion of a Circular Cylinder

Let a circular cylinder of radius a is moving in an infinite mass of liquid at rest at infinity, with velocity U in the direction of x-axis. To find the velocity potential ϕ that will satisfy the given boundary conditions, we have the following conditions :

(*i*) ϕ satisfies the Laplace's equation

$$\nabla^2\phi = 0$$

at every point of the liquid. In polar co-ordinates (r, θ) in two dimensions, $\nabla^2\phi = 0$ takes the form

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{14}$$

which has solutions of the form

$$r^n \cos n\theta, \quad r^n \sin n\theta,$$

where n is any integer, positive or negative. Hence the sum of any number of terms of the form

$$A_n r^n \cos n\theta,\ B_n r^n \sin n\theta$$

is also a solution of (14).

(*ii*) Normal velocity at any point of the cylinder = Velocity of the liquid at that point in that direction, i.e.,

$$-\frac{\partial\phi}{\partial r} = U\cos\theta \qquad \text{when } r = a. \tag{15}$$

(*iii*) Since the liquid is at rest at infinity, velocity must be zero there. Thus,

$$-\frac{\partial\phi}{\partial r} = 0 \text{ and } -\frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \text{ at } r = \infty. \tag{16}$$

The above considerations suggest that we must assume the following suitable form of ϕ.

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta. \tag{17}$$

From (17)

$$-\frac{\partial\phi}{\partial r} = -\left(A - \frac{B}{r^2}\right)\cos\theta, \tag{18}$$

so that using (15), we get

$$U\cos\theta = -\left(A - \frac{B}{a^2}\right)\cos\theta, \text{ valid for all values of } \theta.$$

Hence,

$$-U = \left(A - \frac{B}{a^2}\right).$$

Again the first condition of (16) gives $A = 0$. Thus $B = Ua^2$.
Hence (17) reduces to

$$\phi = \frac{Ua^2}{r}\cos\theta. \tag{19}$$

It may be noted that (19) also satisfies the second condition given by (16). Hence (19) gives the required velocity potential. But

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{Ua^2}{r^2}\sin\theta$$

After integrating, we obtain

$$\psi = -\frac{Ua^2}{r}\sin\theta \tag{20}$$

which gives the stream function of the motion. The complex potential w is given by

$$w = \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = \frac{Ua^2}{z} \tag{21}$$

where $z = re^{i\theta}$.

1.4 Fixed Circular Cylinder in a Uniform Stream

Let a circular cylinder be fixed at the origin and x-axis be chosen in the opposite direction of the stream U. Let R′ be the region $r \geq a$. Now the velocity potential ϕ satisfies the relation

$$\nabla^2 \phi = 0 \text{ in } R'. \tag{22}$$

The boundary conditions are

$$\phi \sim Ux \text{ at infinity,}$$

and

$$-\frac{\partial\phi}{\partial r} = 0 \text{ on the boundary of cylinder.}$$

Let us take

$$\phi = Ur\cos\theta + \phi_1, \tag{23}$$

where ϕ_1 is the contribution due to pressure of the cylinder.

The boundary conditions give

$$\phi_1 \to 0 \text{ at infinity} \tag{24}$$

and

$$-\frac{\partial\phi_1}{\partial r} = U\cos\theta \text{ on } C : r = a. \tag{25}$$

Now, since ϕ is harmonic, so ϕ_1 is harmonic and its normal derivative is prescribed on the boundary.

Now let us assume ϕ_1 to be of the form

$$\phi_1 = \left(Ar + \frac{B}{r}\right)\cos\theta.$$

To satisfy the condition (24), we have A = 0 and from (95), we get $B = a^2U$. Hence

$$\phi = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta. \qquad (26)$$

Again, we have

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

which gives

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta. \qquad (27)$$

Hence, the complex potential is

$$w(z) = Uz + \frac{Ua^2}{z} \text{ in } R'. \qquad (28)$$

The equation of stream line is

$$\psi = \text{constant}$$

or,

$$\left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$

or,

$$\left(y - \frac{a^2 y}{x^2 + y^2}\right) = \text{constant}. \qquad (29)$$

Complex velocity is given by

$$-\frac{dw}{dz} = -V_0\left(1 - \frac{a^2}{z^2}\right) \qquad (30)$$

Then $\frac{dw}{dz} = 0$ implies

$$z = \pm a.$$

Therefore $z = \pm a$ are stagnation points (a point where the velocity is zero is called a stagnation point. The stream lines are not well-defined thereat; a stream line may divide into two branches at such a point).

1.5 Circulation About a Circular Cylinder

If A and P be any two points in a liquid, then $\int_A^P (udx + vdy + wdz)$ is called the **flow** along the path from A to P, where u, v, w are velocity components. If the velocity potential ϕ exists, i.e. if the motion be irrotational, then

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}$$

and so

$$\text{flow} = -\int_A^P \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = \phi_A - \phi_P$$

The flow round a closed curve C is known as **circulation** which is usually denoted by Γ. Thus

$$\Gamma = -\oint_C (udx + vdy + wdz).$$

If the motion is irrotational and the velocity potential ϕ is single-valued, then circulation round C is zero.

Let k be the constant circulation about the cylinder. Then the suitable form of ϕ in two dimensions (r, θ) may be obtained by equating to k the circulation round a circle of radius r. Thus, we have

$$\left(-\frac{\partial \phi}{r \partial \theta} \right)(2\pi r) = k,$$

integrating this we get

$$\phi = -\frac{k\theta}{2\pi}.$$

Again,

$$\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{r \partial \theta},$$

which gives

$$\psi = \frac{k}{2\pi} \ln r,$$

Thus the complex potential due to the circulation about a circular cylinder is given by

$$w = \frac{ik}{2\pi}(\ln r + i\theta)$$

or,

$$w = \frac{ik}{2\pi}\ln z, \text{ (since } z = re^{i\theta}.) \tag{31}$$

1.6 Steaming and Circulation About a Fixed Circular Cylinder

We know that the complex potential w_1 due to the circulation of strength k about the cylinder is given by

$$w_1 = \frac{ik}{2\pi}\ln z.$$

Also, the complex potential w_2 for streaming past a fixed circular cylinder of radius a, with velocity U in the negative direction of x-axis is given by

$$w_2 = Uz + \frac{Ua^2}{z}.$$

Thus, the complex potential w due to the combined effects at any point z is given by

$$w = w_1 + w_2$$

$$= U\left(z + \frac{a^2}{z}\right) + \frac{ik}{2\pi}\ln z. \tag{32}$$

$$\text{i.e. } \phi + i\psi = U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) + \frac{ik}{2\pi}\ln(re^{i\theta})$$

Equating real and imaginary parts, we obtain

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{k\theta}{2\pi} \tag{33}$$

and

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{k}{2\pi}\ln r.$$

Since the velocity will be tangential only at the boundary of the cylinder, so $-\left(\frac{\partial\phi}{\partial r}\right) = 0$ and hence the magnitude of the velocity $\bar{q}$ is given by

$$\left| -\frac{\partial \phi}{r\partial \theta} \right| = \left| 2\,U\sin\theta + \frac{k}{2\,\pi a} \right|.$$

If there is no circulation, i.e. if $k = 0$ there would be points of zero velocity on the cylinder at $\theta = 0$ and $\theta = \pi$, the former being the point at which the incoming stream divides. However, in the presence of circulation, the stagnation points are given by $q = 0$, i.e.

$$\sin\theta = -\frac{k}{4\,\pi Ua}$$

and such points exist when

$$|\,k\,| < 4\,\pi Ua\,. \tag{34}$$

We now determine the pressure at points of the cylinder. The pressure is given by Bernoulli's equation

$$\frac{p}{\rho} = C(\,t\,) - \frac{1}{2}q^{2}. \tag{35}$$

Let Π be the pressure at infinity where the velocity is U and so

$$\frac{\Pi}{\rho} = C(\,t\,) - \frac{1}{2}U^{2}.$$

Then from (35) we obtain

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{2}(\,U^{2} - q^{2}\,)$$

or,

$$p = \Pi + \frac{1}{2}\rho U^{2} - \frac{1}{2}\rho\left(\,2\,U\sin\theta + \frac{k}{2\,\pi a}\,\right)^{2}. \tag{36}$$

If X, Y be the components of the thrust on the cylinder, we have

$$X = -\int_{0}^{2\pi} p\cos\theta a d\theta,$$

$$Y = -\int_{0}^{2\pi} p\sin\theta a d\theta.$$

Using (36) we get $X = 0$, $Y = \rho kU$, showing that the cylinder experiences an upward lift. This effect may be attributed to circulation phenomenon.

1.7 Equation of Motion of a Circular Cylinder

Let a circular cylinder is moving in a liquid at rest at infinity. To calculate the forces acting on the cylinder owing to the pressure of the fluid, we suppose that U, V are the components of the velocity of the cylinder when the center of the cross-section O is (x_0, y_0). Then we have

$$U = \dot{x}_0 \text{ and } V = \dot{y}_0.$$

Let $z_0 = x_0 + iy_0$ and $z - z_0 = re^{i\theta}$ where r denotes the distance from the axis of the cylinder.

On the surface of the cylinder r = a, we must have, the velocity of the liquid normal to the cylinder = normal velocity of the cylinder, i.e.

$$-\frac{\partial \phi}{\partial r} = U\cos\theta + V\sin\theta \text{ at } r = a. \tag{37}$$

Since the liquid is at rest at infinity,

$$-\frac{\partial \phi}{\partial r} = 0 \text{ as } r \to \infty. \tag{38}$$

The conditions (37) and (38) suggest that ϕ is to be taken in the form

$$\phi = \left(Ar + \frac{B}{r} \right)\cos\theta + \left(Cr + \frac{D}{r} \right)\sin\theta. \tag{39}$$

Therefore

$$\frac{\partial \phi}{\partial r} = \left(A - \frac{B}{r^2} \right)\cos\theta + \left(C - \frac{D}{r^2} \right)\sin\theta.$$

Using (37) and (38) we get

$$U = \frac{B}{a^2} - A,\ V = \frac{D}{a^2} - C,\ A = C = 0$$

Thus we have $B = a^2U$, $D = a^2V$,

Hence from (39), the expression for ϕ is given by

$$\phi = \frac{a^2}{r}(U\cos\theta + V\sin\theta). \tag{40}$$

Noting that

$$\frac{\partial \psi}{\partial r} = -\frac{\partial \phi}{r\partial \theta},$$

and using (40) and then integrating this equation, we obtain

$$\psi = \frac{a^2}{r}(-U\sin\theta + V\cos\theta). \tag{41}$$

Hence the complex potential is given by

$$w = \phi + i\psi = \frac{a^2 e^{-i\theta}}{r}(U + iV),$$

i.e.

$$w = \frac{a^2 (U + iV)}{z - z_0}. \tag{42}$$

Now

$$\frac{\partial\phi}{\partial t} + i\frac{\partial\psi}{\partial t} = \frac{\partial w}{\partial t} = \frac{a^2(\dot{U} + i\dot{V})}{z - z_0} + \frac{a^2(U + iV)^2}{(z - z_0)^2}. \tag{43}$$

Equating real parts, we obtain

$$\frac{\partial\phi}{\partial t} = \frac{a^2}{r}(\dot{U}\cos\theta + \dot{V}\sin\theta) + \frac{a^2}{r^2}\left[(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right]. \tag{44}$$

The magnitude of the velocity $\bar{q}$ is given by

$$q^2 = \left|\frac{dw}{dz}\right|^2 = \left|-a^2\frac{U + iV}{(z - z_0)^2}\right|^2 = \frac{a^4(U^2 + V^2)}{r^4}. \tag{45}$$

Omitting the external forces, the pressure at any point is given by Bernoulli's equation as

$$\frac{p}{\rho} = C(t) + \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2,$$

which, on using (44) and (45) gives

$$\frac{p}{\rho} = C(t) + \frac{a^2}{r}\left(\dot{U}\cos\theta + \dot{V}\sin\theta\right) + \frac{a^2}{r^2}\left[(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right] - \frac{1}{2}\frac{a^4}{r^4}(U^2 - V^2) \tag{46}$$

Let p_1 be the pressure at a point (a, θ) on the boundary of the cylinder. Then p_1 is given by (46) on putting $r = a$ as

$$p_1 = \rho C(t) + \rho a\left(\dot{U}\cos\theta + \dot{V}\sin\theta\right) + \rho\left[(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right] - \frac{1}{2}(U^2 + V^2). \tag{47}$$

Let X and Y be the components of force on the cylinder due to fluid thrusts. Then, we have

$$X = -\int_0^{2\pi} ap_1 \cos\theta \, d\theta ,$$

$$Y = -\int_0^{2\pi} ap_1 \sin\theta \, d\theta ,$$

which, with the help of (47), give

$$X = -\rho a^2 \int_0^{2\pi} \dot{U} \cos^2 \theta \, d\theta$$

$$= -\pi a^2 \rho \dot{U}$$

$$= -M'\dot{U} ,$$

where $M' = \pi a^2 \rho$ = the mass of the liquid displaced by the cylinder of unit length.

Similarly,

$$Y = -\pi a^2 \rho \dot{V} = -M'\dot{V} .$$

Corollary :

To show that the effect of the pressure of the liquid is to reduce the extraneous forces in the ratio

$$(\sigma - \rho) : (\sigma + \rho)$$

where σ, ρ are the densities of the cylinder and liquid respectively, we proceed as follows :

Let M be the mass of the cylinder per unit length and X′, Y′ be the components of the extraneous force on the cylinder if there were no liquid. Also let f_x be the acceleration of the extraneous force in x-direction. Then, due to presence of liquid, the resultant force in x-direction is

$$= \pi a^2 \sigma f_x - \pi a^2 \rho f_x$$

$$= \frac{\sigma - \rho}{\sigma} (\pi a^2 \sigma f_x)$$

$$= \frac{\sigma - \rho}{\sigma} X' .$$

Thus the equation of motion in x-direction is of the form

$$M\dot{U} = -M'\dot{U} + \frac{\sigma - \rho}{\sigma} X'$$

or,

$$M\dot{U} = \frac{M}{M+M'} \cdot \frac{\sigma-\rho}{\sigma} X'$$

or,

$$M\dot{U} = \frac{\pi a^2 \sigma}{\pi a^2 \sigma + \pi a^2 \rho} \cdot \frac{\sigma - \rho}{\sigma} X'.$$

Therefore

$$M\dot{U} = \frac{\sigma-\rho}{\sigma+\rho} X'.$$

Similarly,

$$M\dot{V} = \frac{\sigma-\rho}{\sigma+\rho} Y'.$$

Hence the effect of the pressure of the liquid is to reduce the external force in the ratio

$$(\sigma - \rho) : (\sigma + \rho).$$

1.8 Two Coaxial Circular Cylinders

We now determine the velocity potential and the stream function at any point of a liquid contained between two coaxial circular cylinders of radii a and b($a < b$). Let the cylinders are moved suddenly parallel to themselves in directions at right angles with velocities U and V respectively

Then if ϕ be the velocity potential and ψ the stream function at any point (r, θ) in the liquid, then the boundary conditions for the velocity potential ϕ are :

$$-\frac{\partial \phi}{\partial r} = U \cos \theta, \text{ when } r = a$$

and

$$-\frac{\partial \phi}{\partial r} = V \sin \theta, \text{ when } r = b. \tag{48}$$

Now ϕ must satisfy the Laplace's equation

$$\frac{\partial^2 \phi}{\partial^2 r} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial^2 \theta} = 0. \tag{49}$$

at every point of the liquid.

Since (49) has solutions of the form $r^n \cos n\theta$, $r^n \sin n\theta$, where n is any positive or negative integer, the sum of any number of terms of the form $A_n r^n \cos n\theta$, $B_n r^n \sin n\theta$ is also solution of (49). However, a suitable form of ϕ satisfying the given conditions is

$$\phi = \left(Ar + \frac{B}{r} \right)\cos\theta + \left(Cr + \frac{D}{r} \right)\sin\theta. \tag{50}$$

Using the two boundary conditions (48) we obtain for any values θ

$$A - \frac{B}{a^2} = U,\ C - \frac{D}{a^2} = 0,\ \text{and}\ A - \frac{B}{b^2} = 0,\ C - \frac{D}{b^2} = -V.$$

These give

$$A = \frac{Ua^2}{(b^2 - a^2)},\ B = \frac{Ua^2 b^2}{(b^2 - a^2)},\ C = -\frac{Vb^2}{(b^2 - a^2)},\ D = -\frac{Va^2 b^2}{(b^2 - a^2)}.$$

Thus

$$\phi = \frac{Ua^2}{(b^2 - a^2)}\left(r + \frac{b^2}{r} \right)\cos\theta - \frac{Vb^2}{(b^2 - a^2)}\left(r + \frac{a^2}{r} \right)\sin\theta. \tag{51}$$

Since

$$\frac{\partial\phi}{\partial r} = \frac{\partial\psi}{r\partial\theta},$$

we get by using (51)

$$\psi = \frac{Ua^2}{(b^2 - a^2)}\left(r - \frac{b^2}{r} \right)\sin\theta + \frac{Vb^2}{(b^2 - a^2)}\left(r - \frac{a^2}{r} \right)\cos\theta. \tag{52}$$

1.9 The Milne-Thomson's Circle Theorem

Statement : *Let f(z) be the complex velocity potential for the two-dimensional irrotational flow of an incompressible inviscid fluid having no rigid boundaries and such that there are no singularities of flow within the circle* $|z| = a$. *Then, on* *introducing the solid circular cylinder* $|z| = a$ *into the flow, the new complex velocity potential is given by* $w = f(z) + \bar{f}(a^2/z)$ *for* $|z| \geq a$.

Proof : Since the singularities of f(z) occur in the region $|z| > a$, so the singularities of $f(a^2/z)$ lie in $|z| < a$. Hence the singularities of $\bar{f}(a^2/z)$ also lie in $|z| < a$. Thus f(z) and $f(z) + \bar{f}(a^2/z)$ both have the same singularities in the region $|z| > 0$ and, therefore, both functions, considered as complex velocity potentials, may be ascribed to the same hydrodynamical distributions in the region $|z| > a$.

Now, on the circle $|z| = a$, we take $z = ae^{i\theta}$, so that $a^2/z = ae^{-i\theta}$ and, therefore,

$$w = f(z) + \bar{f}(a^2/z) = f(ae^{i\theta}) + \bar{f}(ae^{-i\theta}) = f(ae^{i\theta}) + \overline{f(ae^{i\theta})}$$

Thus, on the circle $|z| = a$, w is the sum of a complex quantity and its complex conjugate and is, therefore, w is a real number, i.e. $\psi = \text{Im}(w) = 0$ on $|z| = a$. Hence, the circular boundary is a stream line across which no fluid flows. We, therefore, conclude that $|z| = a$ is a possible boundary for the new flow for which $w = f(z) + \bar{f}(a^2/z)$ is the appropriate complex velocity potential.

Applications of circle theorem :

Example 1. Uniform flow post a stationary cylinder

We have already seen in Section-1.4 that a uniform stream having velocity –U along the negative direction of x-axis gives rise to a complex potential Uz. Thus, if we take $f(z) = Uz$, then $\bar{f}(a^2/z) = \frac{Ua^2}{z}$. Thus on introducing the circular section $|z| = a$ into the stream, the complex potential for the region $|z| \geq a$ is given by

$$w = f(z) + \bar{f}(a^2/z) = U\left(z + \frac{a^2}{z}\right).$$

If $z = re^{i\theta}$ and $w = \phi + i\psi$, then

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right),\ \psi = U\sin\theta\left(r - \frac{a^2}{r}\right)$$

which are the results obtained in Section—1.4.

Example 2. Uniform stream at incidence with the positive x-axis

The complex potential for such a stream of velocity U is $Uze^{-i\beta}$. Thus, if we take f(z)

$= Uze^{-i\beta}$, then $\bar{f}\left(\frac{a^2}{z}\right) = Ue^{i\beta}.\frac{a^2}{z}$. Hence, when the cylinder of section $|z| = a$ is introduced, the complex potential in $|z| \geq a$ becomes $w = U\left\{ze^{-i\beta} + \left(\frac{a^2}{z}\right)e^{i\beta}\right\}$.

1.10 Theorem of Blasius

Statement : *Suppose that, in a steady two-dimensional irrotational motion given by the relation $w = f(z)$, i.e. $\phi + i\psi = f(\alpha + iy)$, the hydrodynamical pressures on the contour of a fixed cylinder are (X, Y) and a couple N about the origin of coordinates. Then*

$$X - ir = \frac{1}{2} i\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz,$$

and

$$M = Re\left\{-\frac{1}{2}\rho \oint_C z\left(\frac{dw}{dz}\right)^2 dz\right\} \tag{53}$$

where ρ is the density and the integrations are taken round any contour surrounding the cylinder.

Proof : Let the normal to the cylinder at the point P(x, y) make an angle θ with the positive direction of x-axis.

Then, for the action on the arc ds and P, we have

$$dX = -p\sin\theta\, ds,\ dY = p\cos\theta\, ds$$

$$\text{i.e. } dX = -pdy,\ dY = pdx$$

so that

$$X = \oint_{C'} pdy,\ Y = \oint_{C'} pdx.$$

and, therefore,

$$X - iY = -i\oint_{C'} p(dx - idy).$$

where the integrals are round the contour C' of the cylinder.

Since there is no external force and the fluid is moving irrotationally and steadily, so the pressure equation is given by

$$\frac{p}{q}+\frac{1}{2}q^2 = \text{constant} = A.$$

Thus

$$X - iY = -i\oint_{C'}\rho\left(A-\frac{1}{2}q^2\right)(dx - idy)$$

$$= \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)^2 d\bar{z}$$

$$= \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)\left(\overline{\frac{dw}{dz}}\right)d\bar{z}$$

$$= \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)^2 d\bar{w}.$$

Now the contour of the cylinder is a stream line, i.e. on C', ψ = constant. Also

$$dw = d\bar{w}.$$

Therefore

$$X - iY = \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)^2 dz.$$

Now in the plane outside the cylinder, it may be possible to have singularity in the function $\left(\frac{dw}{dz}\right)^2$ if there is any physical singularity in the fluid (such as a source or a vortex). Thus, if we take a larger countour C surrounding C' such that there are no singularities between C and C'; or more generally, if such singularities exist, then the sum of the residues of $\left(\frac{dw}{dz}\right)^2$ at all poles between C and C' is zero, then the integrals of this function have the same value for all such contours and we have

$$X - iY = \frac{1}{2}i\rho\oint_C\left(\frac{dw}{dz}\right)^2 dz.$$

Again

$$N = \oint_{C'}(pxdx + pydy)$$

$$= \text{Real part of } \oint_{C'} p(x+iy)(dx - idy)$$

$$= \text{Real part of } \oint_{C'} pz d\bar{z}$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right)^2 d\bar{z} \right]$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right) d\bar{w} \right]$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right) dw \right]$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right)^2 dz \right].$$

Considering the same limitation as before regarding singularities in the liquid, the integral may be taken round any contour C which surrounds the cylinder.

1.11 Transformations or Mapping

The set of equations

$$u = u(x, y), \; v = v(x, y) \tag{54}$$

defines, in general, a transformation or mapping which establishes a correspondence between points in the uv- and xy-planes. The equations (54) are called transformation equations. If to each point of the uv-plane there corresponds one and only one point of the xy plane and conversely, we speak of a one-to-one transformation or mapping.

Conformal mapping

Suppose that under the transformation (54), the point (x_0, y_0) of the xy-plane is mapped into the point (u_0, v_0) of the uv-plane while curves C_1 and C_2 [intersecting at (x_0, y_0)] are mapped respectively into curves C'_1 and C'_2. Then, if the transformation is such that the angle at (x_0, y_0) between C_1 and C_2 is equal to the angle at (u_0, v_0) between C''_1 and C''_2 both in magnitude and sense, the transformation or mapping is said to be *conformal* at (x_0, y_0). A mapping which preserves the magnitudes of angles but not necessarily the sense is called *isogonal.*

1.12 The Schwarz-Christoffel Transformations

Any simple closed polygon with n vertices in the z-plane (z = x + iy) can be transformed into the real axis in the $\zeta\,(=\xi+i\eta)$-plane, the interior points of the polygon corresponding to points on one side of the real axis $\eta = 0$, the transformation-effective relation being

$$\frac{dz}{d\zeta} = A(\zeta-a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta-a_2)^{\frac{\alpha_2}{\pi}-1}\cdots(\zeta-a_n)^{\frac{\alpha_n}{\pi}-1} \tag{55}$$

or, $$z = A\int(\zeta-a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta-a_2)^{\frac{\alpha_2}{\pi}-1}\cdots(\zeta-a_n)^{\frac{\alpha_n}{\pi}-1}\,d\zeta+B \tag{56}$$

where A and B are constants which may be complex, $\alpha_1, \alpha_2, ..., \alpha_n$ are the interior angles of the polygon and $a_1, a_2, ..., a_n$ are the points on the real axis $\eta = 0$ that correspond to the angular points of the polygon in the z-plane.

The following facts should be noted :

1. Any three points of $a_1, a_2, ..., a_n$ can be chosen at will.
2. The constants A and B determine the size, orientation and position of the polygon.
3. It is convenient to choose one point, say a_n at infinity in which case the last factor of (54) and (55) involving a_n is not present.
4. Infinite open polygons can be considered as limiting case of closed polygons.

1.13 Elliptic Coordinates

Let

$$z = c\cosh\zeta,\ \text{where } z = x + iy,\ \zeta = \xi + i\eta.$$

Then $x + iy = c\cosh(\xi + i\eta) = e(\cosh\xi\cos\eta + i\sinh\xi\sin\eta)$

so that $x = c\cosh\xi\cos\eta\ \ y = c\sinh\xi\sin\eta$. (56)

Obviously $$\frac{x^2}{c^2\cosh^2\xi}+\frac{y^2}{c^2\sinh^2\xi}=1 \tag{57}$$

and $$\frac{x^2}{c^2\cos^2\eta}+\frac{y^2}{c^2\sin^2\eta}=1. \tag{58}$$

Thus ξ = const. and η = const. represent confocal ellipses and hyperbolas respectively, the distance between the focii being 2c.

Let a, b be the semi-axes of the ellipse (57). Then for $\xi = \alpha$,

$$a = c\cosh\alpha,\ b = c\sinh\alpha,\ c^2 = a^2 - b^2$$

and

$$a + b = ce^{\alpha},\ a - b = ce^{-\alpha},\ \alpha = \frac{1}{2}\log\frac{a+b}{a-b}.$$

The parameters ξ, η are called **elliptic coordinates.**

1.14 The Joukowski Transformations

The transformation

$$z = \zeta + \frac{c^2}{4\zeta} \tag{59}$$

is one of the simplest and most important transformations of two-dimensional motion. By means of this transformation we can map the ζ-plane on the z-plane, and vice versa. From (59), it can be shown that when $|z|$ is large, we have $\zeta = z$ nearly, so that the distant parts of the two-planes are unaltered. Thus a uniform stream at infinity in the z-plane will correspond to a uniform stream of the same strength and direction in the ζ-plane.

We now consider the inverse transformation of (59), viz. $\zeta = \frac{1}{2}\left(z \pm \sqrt{z^2 - c^2}\right)$, or confining to positive sign only,

$$\zeta = \frac{1}{2}\left(z + \sqrt{z^2 - c^2}\right) \tag{60}$$

It can be readily shown that the region outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is mapped into the region outside the circle $|\zeta| = \frac{1}{2}(a+b)$.

Application. Streaming past a fixed elliptic cylinder

Let us consider the stream whose complex potential is $U\zeta e^{-i\beta}$ in the ζ-plane. Then, on inserting the circular cylinder $|\zeta| = \frac{1}{2}(a+b)$ into the stream, the new complex potential is given by circle theorem as

$$w_1 = U\zeta e^{-i\beta} + \frac{U(a+b)^2}{4\zeta} e^{i\beta} \tag{61}$$

Now by Joukowski's transformation (60), the region outside the circle $|\zeta| = \frac{1}{2}(a+b)$ is mapped on the region outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence the complex potential w for the flow past a fixed elliptic cylinder can be obtained from (60) and (61) by eliminating ζ as

$$w = \frac{1}{2}U\left[e^{-i\beta}\left(z + \sqrt{z^2 - c^2}\right) + \frac{(a+b)^2 e^{i\beta}}{z + \sqrt{z^2 - c^2}}\right]$$

Using the transformation $z = c\cosh\zeta$ for elliptic coordinates, we have $\sqrt{z^2 - c^2} = c\sinh\zeta$ and so $z + \sqrt{z^2 - c^2} = ce^{\zeta}$. Thus

$$w = \frac{1}{2}U\left[e^{-i\beta} \cdot ce^{\zeta} + \frac{(a+b)^2}{ce^{\zeta}} e^{i\beta}\right]$$

$$= \frac{1}{2}U\sqrt{\frac{a+b}{a-b}}\left[(a-b)e^{\zeta - i\beta} + (a+b)e^{-\zeta + i\beta}\right]$$

Hence on the ellipse $\xi = \alpha$, whence $a + b = ce^{\alpha}$ and $a - b = ce^{-\alpha}$, we get

$$w = \frac{1}{2}U(a+b)\left[e^{\zeta - i\beta - \alpha} + e^{(\zeta - i\beta - \alpha)}\right]$$

i.e. $$w = U(a+b)\cos h(\zeta - i\beta - \alpha). \tag{62}$$

This is the required complex potential for the streaming past a fixed elliptic cylinder.

In particular, if the stream were parallel to the real axis, so that $\beta = 0$, then

$$w = U(a+b)\cos h(\zeta - \alpha). \tag{63}$$

As a special case, we impart to the whole system a velocity U inclined at an angle β with the x-axis. Then the stream is reduced to rest and the cylinder moves with velocity U, so that the complex potential is

$$w = \frac{U(a+b)^2}{4\zeta} e^{i\beta} = \frac{U(a+b)^2}{2(z + \sqrt{z^2 - c^2})} e^{i\beta} = \frac{U(a+b)^2}{2c} e^{-\zeta + i\beta}$$

$$= \frac{U(a+b)}{2} e^{-\zeta + i\beta + \alpha}. \tag{64}$$

This is the complex potential for the elliptic cylinder moving in an infinite liquid with velocity U inclined at an angle β with the x-axis. In particular, if the elliptic cylinder moves parallel to the x-axis, so that $\beta = 0$, then

$$w = \frac{1}{2} U (a + b) e^{-\zeta + \alpha}. \tag{65}$$

1.15 The Aerofoil

The aerofoil used in modern aeroplanes has a profile of "fish" type, indicated in figure. Such an aerofoil has a blunt leading edge and a sharp trailing edge. The projection of the profile on the double tangent, as shown in the diagram, is the chord. The ratio of the span to the chord is the aspect ratio.

The camber line of a profile is the locus of the point midway between the points in which an ordinate perpendicular to the chord meets the profile. See figure 2.15

The camber is the ratio of the maximum ordinate of the flow round such an aerofoil on the following assumptions :

1. That the air behaves as an incompressible inviscid fluid.
2. That the aerofoil is a cylinder whose cross-section is a curve of the above type.
3. That the flow is two-dimensional irrotational cyclic motion.

The above assumptions are of course only approximations to the actual state of affairs, but by making these simplifications it is possible to arrive at a general understanding of the principles involved.

It has been found that profiles obtained by conformal transformation of circle by the simple Joukowski transformation make good wing shapes, and that the lift can be calculated from the known flow with respect to a circular cylinder.

1.16 The Theorem of Kutta and Joukowski

Statement : *If an aerofoil of any shape be placed in a uniform wind of speed V, then the resultant thrust on the aerofoil is a lift of magnitude $k\rho v$ per unit length and is at right angles to the wind, where k is the circulation round the cylinder.*

Direction of flight

Figure 2.15

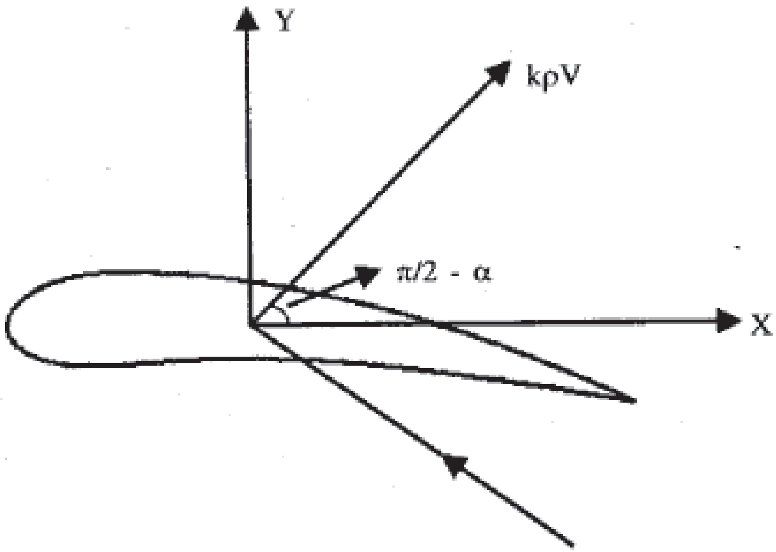

Figure 2.16

Proof. Since there is a uniform wind, the velocity at a great distance from the aerofoil must tend to the wind velocity, and therefore if $|z|$ is sufficiently large, so that we may write

$$-\frac{dw}{dz} = -Ve^{i\alpha} + \frac{A}{z} + \frac{B}{z^2} + \cdots \qquad (66)$$

where α is the angle of incidence or angle of attack.

Thus

$$w = Vze^{i\alpha} - A\ln z + \frac{B}{z} + \cdots$$

and since there is circulation k, we must have

$$-A = \frac{ik}{a\pi}, \qquad (67)$$

for ln z increases by $2\pi i$ when we go once round the aerofoil in the positive sense.

From (66) and (67) we get,

$$\left(\frac{dw}{dz}\right)^2 = V^2 e^{2i\alpha} + \frac{ikV}{\pi z} e^{i\alpha} - \frac{k^2 + 8\pi^2 BVe^{i\alpha}}{4\pi^2 z^2} - \cdots. \tag{68}$$

If we now integrate round a circle whose radius is sufficiently large for the expression (68) to be valid, the theorem of Blasius gives

$$X - iY = \left(\frac{1}{2} i\rho\right) 2\pi i \left(\frac{ikVe^{i\alpha}}{\pi}\right)$$

$$= -ik\rho Ve^{i\alpha}$$

so that, changing the sign of i we obtain

$$X + iY = k\rho Ve^{i\left(\frac{1}{2}\pi - \alpha\right)}.$$

Comparison with above figure shows that this force has all the properties stated in the enunciation.

1.17 Motion of an Elliptic cylinder

(i) To determine the velocity potential and stream function when an elliptic cylinder moves in an infinite liquid with velocity U parallel to the axial plane through the major of a cross-section.

For any cylinder moving with velocities U and V parallel to axes and rotating with an angular velocity ω, we know that on the cylinder

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + \text{constant (A, say)}.$$

Here

$$V = 0, \omega = 0.$$

Hence the stream function is given by

$$\psi = -Uy + A. \tag{69}$$

Let the cross-section be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the same as $\xi = \alpha$, if $a = c \cosh \alpha$, $b = \sinh \alpha$ and $c^2 = a^2 - b^2$, where

$$x = c \cosh \xi \cos \eta, \tag{70}$$

and

$$y = c \sinh \xi \sin \eta. \tag{71}$$

Using (70) and (71), (69) becomes

$$\psi = - Uc \sinh \alpha \sin \eta + A. \tag{72}$$

Since ψ contains $\sin \eta$ and the liquid is at rest at infinity, ψ must be of the form $e^{-\xi} \sin \eta$. We therefore, assume that

$$\phi + i\psi = Be^{-(\xi + i\eta)} \tag{73}$$

so that

$$\psi = - Be^{-\eta} \sin \eta. \tag{74}$$

Then at boundary $\xi = \alpha$, we must have for all values of η,

$$A = 0, \; B = Uce^{\alpha} \sinh \alpha.$$

Thus

$$\psi = - Uce^{\alpha-\xi} \sinh \alpha \sin \eta \tag{75}$$

is a stream function which will make the boundary of the ellipse a stream line, when the cylinder moves with velocity U.

But

$$c \sinh \alpha = b \text{ and } e^{\alpha} = \left(\frac{a+b}{a-b} \right)^{\frac{1}{2}}. \tag{76}$$

Using (75) and (76), (7) can be written in the form

$$\psi = - Ub \left(\frac{a+b}{a-b} \right)^{\frac{1}{2}} e^{-\xi} \sin \eta. \tag{77}$$

Also from (75),

$$\phi = Ub \left(\frac{a+b}{a-b} \right)^{\frac{1}{2}} e^{-\xi} \cos \eta. \tag{78}$$

Hence we obtain

$$w = \phi + i\psi = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}. \tag{79}$$

(ii) To determine the velocity potential and the stream function when an elliptic cylinder moves in an infinite liquid with velocity V parallel to the axial plane through the minor axis of a cross-section.

Proceeding as in (i), we can obtain

$$\phi = Va\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \cos\eta, \tag{80}$$

$$\psi = Va\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin\eta, \tag{81}$$

and

$$w = iVa\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}. \tag{82}$$

(iii) To determine the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity v in a direction making an angle β with the major axis of the cross section of the cylinder.

The components of v along coordinate axes are

$$U = v\cos\beta$$

and

$$V = v\sin\beta$$

Let w_1 and w_2 be the complex potentials corresponding to the motion of the cylinder with velocities U and V respectively. Then from (79) and (82), we obtain

$$w_1 = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}$$

$$= bv\cos\theta\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)},$$

and

$$w_2 = iva\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}$$

$$= iav\sin\theta\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}.$$

Hence the complex potential due to velocity v is given by

$$w = w_1 + w_2$$

$$= cv\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\zeta}\sinh(\alpha+i\beta),$$

where $\zeta = \xi + i\eta$, $b = c\sinh\alpha$, $a = c\cosh\alpha$. Thus

$$w = v(a+b)e^{-\zeta}\sinh(\alpha+i\beta), \text{ since } c^2 = a^2 - b^2.$$

1.18 Liquid Streaming Past a Fixed Elliptic Cylinder

To determine ϕ and ψ for a liquid streaming past a fixed elliptic cylinder with velocity U parallel to major axis of the section.

Superimpose a velocity U on the cylinder and on liquid both in the sense opposite to the velocity of the liquid. This brings the liquid at rest and the cylinder in motion with velocity U. Hence, some suitable term must be added to each of the expressions for ϕ and ψ obtained in (69) of Art. **1.17**. When the stream flows from positive x-axis to negative x-axis, we have

$$-\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y} = -U. \tag{83}$$

Accordingly, we must add a term Ux to ϕ and Uy to ψ as obtained in Art. **1.17**. Thus, we have

$$\phi = Ux + Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi}\cos\eta$$

$$= U(a^2-b^2)^{\frac{1}{2}}\cosh\xi\cos\eta + Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi}\cos\eta, \tag{84}$$

and

$$\psi = Uy - Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi}\sin\eta$$

$$= U(a^2 - b^2)^{\frac{1}{2}} \sinh\xi \sin\eta - Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin\eta. \quad (85)$$

Then the complex potential is given by

$$w = Uz + Ube^{\alpha-\zeta}. \quad (86)$$

Another form for ϕ, ψ and w, we can be obtained as

$$\phi = Uce^{\alpha} \cos\eta \cosh(\xi - \alpha), \quad (87)$$

$$\psi = Uce^{\alpha} \sin\eta \sinh(\xi - \alpha), \quad (88)$$

and

$$\omega = U(a+b)\cosh(\zeta - \alpha). \quad (89)$$

which is the result (63) obtained in Section—1.16.

1.19 Rotating Elliptic Cylinder

To determine ϕ and ψ when an elliptic cylinder is rotating with angular velocity ω in an infinite mass of the liquid at rest at infinity.

For any cylinder moving with velocity U and V parallel to axes and rotating with an angular velocity ω, we know that on the cylinder

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + \text{constant, say A}. \quad (90)$$

Let the cross-section be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the same as $\xi = \alpha$, if $a = c \cosh \alpha$, $b = c \sinh \alpha$ and $c^2 = a^2 - b^2$. The elliptic coordinates (ξ, η) are given by

$$x = c\cosh\xi \cos\eta,$$

$$y = c\sinh\xi \sin\eta. \quad (91)$$

Here

$$U = V = 0.$$

So using (91), (90) reduces to

$$\psi = \frac{1}{4}\omega c^2 (\cosh 2\xi + \cos 2\eta) + A. \quad (92)$$

Since, ψ contains $\cos 2\eta$ and the liquid is at rest at infinity, ψ must be taken in the form

$$\psi = Be^{-2\xi} \cos 2\eta \quad (93)$$

and hence

$$\phi = Be^{-2\xi} \sinh 2\eta. \quad (94)$$

Then at the boundary $\xi = \alpha$, we obtain for all values of η

$$B = \frac{1}{4}\omega c^2 e^{2\alpha}$$

and

$$A = -\frac{1}{4}\omega c^2 \cosh 2\alpha.$$

Thus ϕ and ψ reduce to

$$\phi = \frac{1}{4}\omega (a+b)^2 e^{-2\xi} \sin 2\eta, \quad (95)$$

$$\psi = \frac{1}{4}\omega (a+b)^2 e^{-2\xi} \cos 2\eta. \quad (96)$$

Hence the complex potential function is

$$\omega = \frac{1}{4} i\omega (a+b)^2 e^{-2\zeta}, \text{ since } \zeta = \xi + i\eta. \quad (97)$$

1.20 Motion of a Liquid in Rotating Elliptic Cylinders

Let the elliptic cylinder containing liquid rotate with angular velocity ω. The stream function ψ must satisfy the Laplace's equation

$$\nabla^2 \psi = 0$$

and on the boundary it satisfies the condition

$$\psi = \frac{1}{2}\omega (x^2 + y^2) + A. \quad (98)$$

We assume that

$$\psi = B(x^2 - y^2). \quad (99)$$

On the boundary of the cylinder, we must have

$$\frac{x^2}{A\Big/\left(B-\frac{1}{2}\omega\right)}+\frac{y^2}{A\Big/\left(-B-\frac{1}{2}\omega\right)}=1. \tag{100}$$

We also know that the boundary of the cylinder is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{101}$$

Comparing (100) with (101) we get

$$B=\frac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2},$$

so that

$$\psi=\frac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}(x^2-y^2). \tag{102}$$

Then from (99)

$$\phi=-\omega\frac{a^2-b^2}{a^2+b^2}xy. \tag{103}$$

The magnitude of the velocity $\bar{q}$ is given by

$$q^2=\left(-\frac{\partial\phi}{\partial x}\right)^2+\left(-\frac{\partial\phi}{\partial y}\right)^2$$

$$=\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)(x^2+y^2). \tag{104}$$

K.E. of the liquid contained in rotating cylinder is given by

$$T=\frac{1}{2}\rho\iint q^2\,dxdy$$

$$=\frac{1}{8}\pi\rho ab\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}. \tag{105}$$

1.21 Flow Past a Plate

If $b = 0$, our ellipse degenerates into the line joining the foci, namely $\alpha = 0$, and therefor $a = c$. Hence for the flow past a plate inclined at angle θ to the stream, we have

$$\omega = Ua\cosh(\zeta - i\theta)$$

The stagnation points still lie on the hyperbolic branches.

$$\eta = \theta,\ \eta = \pi + \theta.$$

The speed becomes infinite at the edges of the plate, so that the solution cannot represent the complete motion past on an actual plate.

In terms of z, we have

$$\omega = U(z\cos\theta - i\sqrt{z^2 - a^2}\sin\theta).$$

When the plate is perpendicular to the stream, then $\theta = \frac{\pi}{2}$, so that

$$\omega = -iU\sqrt{z^2 - a^2}.$$

1.22 Illustrative Solved Examples

Example 1 :

In the case of two dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is U in a fixed direction, where U is a variable. Show that the maximum value of the velocity at any point of the fluid is 2U. Prove that the force necessary to hold the disc is $2m\dot{U}$, where m is the liquid displaced by disc.

Solution :

The velocity potential for the liquid streaming past a fixed circular disc is given by

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta, \tag{1}$$

where a is the radius of the disc. This gives

$$\frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \text{ and } \frac{\partial\phi}{\partial\theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta$$

Therefore

$$q^2 = \left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = U^2\left(1 - \frac{a^2}{r^2}\right)^2\cos^2\theta + U^2\left(1 + \frac{a^2}{r^2}\right)^2\sin^2\theta$$

$$= U^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right). \tag{2}$$

which is maximum with respect to θ when $\cos 2\theta = -1$ i.e. $2\theta = \pi$ and then

$$q^2 = U^2\left(1+\frac{2a^2}{r^2}+\frac{a^4}{r^4}\right) \quad \text{or} \quad q = U\left(1+\frac{a^2}{r^2}\right)$$

Now q is further maximum with respect to r when r is minimum, i.e., when r = a. Hence the required maximum value of q is given by

$$q = 2U.$$

By Bernoulli's equation, the pressure p is given by

$$\frac{p}{q} = \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2 + F(t). \tag{3}$$

Using (1) and (2), (3) reduces to

$$\frac{p}{\rho} = F(t) - \frac{1}{2}U^2\left(1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right)+\dot{U}\left(r+\frac{a^2}{r}\right)\cos\theta.$$

Putting r = a, the pressure on the boundary of the disc is given by

$$\frac{p}{\rho} = F(t) - 2U^2\sin^2\theta + \dot{U}.2a\cos\theta.$$

Then the resultant pressure on the disc

$$= \int_0^{2\pi}(-p\cos\theta)a\,d\theta = -\rho a\int_0^{2\pi}\left[F(t)-2U^2\sin^2\theta+2\dot{U}a\cos\theta\right]d\theta, \text{ by (4)}$$

$$= -2\rho a^2\dot{U}\int_0^{2\pi}\cos^2\theta\,d\theta = -2\pi a^2\rho\dot{U} = -2m\dot{U} \qquad \text{since } m = \pi a^2\rho$$

Hence the desired force necessary to hold the disc is $2m\dot{U}$.

Example 2 :

A circular cylinder is placed in uniform stream, find the force acting on the cylinder.

Solution :

We know that the complex potential for the undisturbed motion in a uniform stream with velocity components U, V is given by $w = (U + iV)z$. Using Milne-Thomson's circle theorem, the complex potential for the present problem is

$$w = (U - iV)z + (U + iV)(a^2/z)$$

Therefore

$$\frac{dw}{dz} = U - iV - (U + iV)(a^2/z^2)$$

If the pressure thrusts on the contour of the fixed circular cylinder be represented by a force (X, Y) and a couple of moment N about the origin of co-ordinates, then by Blasius' theorem, we have

$$X-iY=\frac{1}{2}i\rho\int_C\left(\frac{d\omega}{dz}\right)^2 dz=\frac{1}{2}i\rho\int\{(U-iV)-(U+iV)(a^2/z^2)\}^2 dz=0$$

so that

$$X=0 \quad \text{and} \quad Y=0-$$

and

$$N=\text{Real part of } -\frac{1}{2}\rho\int_C z\left(\frac{d\omega}{dz}\right)^2 dz$$

$$=\text{real part of } -\frac{1}{2}\rho\int_C z\left\{U-iV-(U-iV)\frac{a^2}{z^2}\right\}^2 dz$$

$$=\text{real part of } -\frac{1}{2}\rho\{-2(U^2+V^2)a^2\}2\pi i=0$$

Therefore X = Y = N = 0, showing that neither a force nor a couple acts on the cylinder.

Example 3 :

A circular cylinder is fixed across a stream of velocity U with a circulation k round the cylinder. Show that the maximum velocity in the liquid is $2U+(k/2\pi a)$, where a is the radius of the cylinder.

Solution :

The velocity potential ϕ for the motion is

$$\phi=U\left(r+\frac{a^2}{r}\right)\cos\theta-\frac{k\theta}{2\pi}, \qquad (1)$$

where r is measured from the centre of the cross-section of the cylinder.

Then the velocity q is given by

$$q^2=\left(-\frac{\partial\phi}{\partial r}\right)^2+\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$$

$$=U^2\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta+\left\{U\left(1-\frac{a^2}{r^2}\right)\sin\theta+\frac{k}{2\pi r}\right\}^2$$

$$= U^2\left(1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right)+\frac{Uk}{\pi r}\left(1+\frac{a^2}{r^2}\right)\sin\theta+\frac{k^2}{4\pi^2 r^2},$$

which is maximum with respect to r when r is minimum, i.e. when r = a.

Thus

$$q^2 = U^2(2-2\cos\theta)+\frac{2Uk}{\pi a}\sin\theta+\frac{k^2}{4\pi^2 a^2}$$

$$=4U^2\sin^2\theta+\frac{2Uk}{\pi a}\sin\theta+\frac{k^2}{4\pi^2 a^2}$$

$$=\left(2U\sin\theta+\frac{k}{2\pi a}\right)^2 \qquad (2)$$

Now q is further maximum with respect to θ when sin θ = 1 i.e. θ = π/2. Thus, from (2) the desired maximum velocity is given by

$$q^2=\left(2U+\frac{k}{2\pi a}\right)^2 \qquad \text{i.e. } q=2U+\frac{k}{2\pi a}.$$

Example 4 :

An infinite elliptic cylinder with semi axes a, b is rotating round its axes with angular velocity ω in an infinite liquid of density ρ which is at rest at infinity. Show that if the fluid is under the action of no force, the moment of the fluid pressure on the cylinder round the center is

$$\frac{1}{8}\pi\rho c^4\frac{d\omega}{dt} \quad \text{where} \quad c^2 = a^2 + b^2$$

Solution :

Using Bernoulli's equation, pressure p at any point is given by

$$\frac{p}{q}=C-\frac{1}{2}q^2+\frac{\partial\phi}{\partial t} \qquad (1)$$

Now for an elliptic cylinder rotating with an angular velocity ω in an infinite fluid, velocity potential ϕ and complex potential ω are given by

$$\phi=\frac{1}{4}\omega(a+b)^2\, e^{-2\xi}\sin 2\eta \qquad (2)$$

and

$$w = \frac{1}{4} i\omega (a+b)^2 e^{-2\xi}, \tag{3}$$

where

$$z = x + iy = c \cosh \zeta \quad \text{and} \quad \zeta = \xi + i\eta. \tag{4}$$

Therefore

$$q^2 = \left|\frac{d\omega}{dz}\right|^2 = \left|\frac{d\omega}{d\zeta}\cdot\frac{d\zeta}{dz}\right|^2 = \left|\frac{1}{4} i\omega (a+b)^2 e^{-2\zeta} (-2)\frac{1}{c\sinh\zeta}\right|^2$$

$$= \frac{\omega^2 (a+b)^4}{4c^2}\left|\frac{e^{-2\xi}.e^{-2i\eta}}{\sinh(\xi+i\eta)}\right|^2$$

$$= \frac{\omega^2 (a+b)^4 e^{-4\xi}}{4c^2} \times \frac{1}{\sinh^2 \xi + \sin^2 \eta} \tag{5}$$

and

$$\frac{\partial\phi}{\partial t} = \frac{1}{4}(a+b)^4 e^{-2\xi} \sin 2\eta \frac{d\omega}{dt} \tag{6}$$

Using (1), (5) and (6), the pressure at any point on the boundary of the ellipse $\xi = \alpha$ is given by

$$\frac{p}{\rho} = C - \frac{\omega^2 (a+b)^2 e^{-4\alpha}}{8c^2 (\sinh^2 \alpha + \sin\eta)} + \frac{1}{4}(a+b)^2 e^{-2\alpha} \sin 2\eta \frac{d\omega}{dt} \tag{7}$$

Now the pressure on an elementary are ds of elliptic boundary at a point P (of eccentric angle η) is pds. Let θ be the angle between tangent and radius vector.

Then from calculus, we have

$$\cos\theta = \frac{dr}{ds} \tag{8}$$

Now the moment of the fluid pressure on the element ds about the center

$$= -\text{ prds } \cos\theta = -\text{ prds}, \qquad \text{by (8)}$$

$$= p.\frac{1}{2}(a^2 + b^2)\sin 2\eta d\eta \qquad \left[\text{since, } rdr = -\frac{1}{2}(a^2 + b^2)\sin 2\eta d\eta\right]$$

Therefore, the required total moment of the liquid pressure on the elliptic cylinder about the centre is

$$= \frac{a^2 - b^2}{2} \int_0^{2\pi} p \sin 2\eta \, d\eta$$

$$= \frac{a^2 - b^2}{2} \rho \int_0^{2\pi} \left[C - \frac{\omega^2 (a+b)^4}{8c^2} \frac{e^{-4\alpha}}{\sinh^2 \alpha + \sin^2 \eta} + \frac{(a+b)^2}{4} e^{-2\alpha} \sin 2\eta \frac{d\omega}{dt} \right] \sin 2\eta \, d\eta$$

$$= \frac{a^2 - b^2}{2} \rho \int_0^{2\pi} \frac{(a+b)^2}{4} e^{-2\alpha} \sin^2 2\eta \frac{d\omega}{dt} \, d\eta \text{ (other integrals vanish)}$$

$$= \frac{(a^2 - b^2)(a+b)^2 e^{-2\alpha}}{8} \rho \frac{d\omega}{dt} \int_0^{2\pi} \sin^2 2\eta$$

$$\frac{c^2 (a+b)^2}{8} \frac{a-b}{a+b} \rho \frac{d\omega}{dt} \int_0^{2\pi} \frac{1 - \cos 4\eta}{2} d\eta \qquad \left[\text{since, } c^2 = a^2 - b^2, \ e^{2\alpha} = \frac{a+b}{a-b} \right]$$

$$= \frac{c^2 (a^2 - b^2)}{8} \rho \frac{d\omega}{dt} \pi = \frac{1}{8} \pi \rho c^4 \frac{d\omega}{dt}.$$

Example 5 :

In the two-dimensional irrotational motion of a liquid streaming past a fixed elliptic disc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the velocity at infinity being parallel to the major axis and equal to U, prove that if

$$x + iy = c \cosh (\xi + i\eta), \quad a^2 - b^2 = c^2 \ \text{ and } \ a = c \cosh \alpha, \ b = c \sinh \alpha,$$

the velocity at any point is given by

$$q^2 = U^2 \frac{a+b}{a-b} \cdot \frac{\sinh^2 (\xi - \alpha) + \sin^2 \eta}{\sinh^2 \xi + \sin^2 \eta}$$

and that it has maximum value $\frac{U(a+b)}{a}$ at the end of the minor axis.

Solution :

The velocity potential for the case "Liquid streaming past a fixed elliptic cylinder" is given by

$$w = U(a + b) \cosh (\xi - \alpha) \qquad (1)$$

Now,

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{dw}{d\zeta}\frac{d\zeta}{dz}\right|.$$

Now,

$$q = \frac{U(a+b)}{c}\left|\frac{\sinh(\zeta-\alpha)}{\sinh\zeta}\right| \quad [\text{using (1) and } z = c\cosh\zeta] \tag{2}$$

But

$$|\sinh(\zeta-\alpha)| = |\sinh(\xi-\alpha+i\eta)|, \quad \text{as } \zeta = \xi + i\eta$$

$$= |\sinh(\xi-\alpha)\cos\eta + i\cosh(\xi-\alpha)\sin\eta|$$

$$= \sqrt{\sinh^2(\xi-\alpha)\cos^2\eta + \cosh^2(\xi-\alpha)\sin^2\eta}$$

$$= \sqrt{\sinh^2(\xi-\alpha) + \sin^2\eta}$$

Similarly,

$$|\sinh\zeta| = \sqrt{\sinh^2\xi + \sin\eta}$$

Since,

$$q = \frac{U(a+b)}{\sqrt{a^2-b^2}}\left[\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}\right]^{1/2} \qquad (\because\ c = \sqrt{a^2+b^2}\,)$$

so that

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\left[\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}\right] \tag{3}$$

(3) gives the required value of velocity.

To determine the maximum value of q, we rewrite (3) as follows :

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\left[1 - \frac{\sinh^2\xi + \sin^2(\xi-\alpha)}{\sinh^2\xi+\sin^2\eta}\right] \tag{4}$$

But $\sinh\xi > \sinh(\xi - \alpha)$. Hence for a given ξ, (4) shows that q will be maximum when $\sin\eta$ is maximum i.e. $\eta = \frac{\pi}{2}$. Then (3) gives

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{1+\sinh^2(\xi-\alpha)}{1+\sinh^2\xi} = U^2\left(\frac{a+b}{a-b}\right)\frac{\cosh^2(\xi-\alpha)}{\cosh^2\xi} \qquad (5)$$

$$= U^2\left(\frac{a+b}{a-b}\right)\left[\frac{\cosh\xi\cosh\alpha - \sinh\xi\sin\alpha}{\cosh\xi}\right]^2$$

$$= U^2\left(\frac{a+b}{a-b}\right)(\cosh\alpha - \tanh\xi\sin\alpha)^2, \qquad (6)$$

showing that q will be maximum when $\tanh\xi$ is minimum i.e. ξ is minimum. Since we have an elliptic cylinder surrounded by liquid, the minimum value of ξ is α. Hence putting $\xi = \alpha$ in (5), the required maximum value of q is given by

$$(q_{max})^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{1}{\cosh^2\alpha} = U^2\left(\frac{a+b}{a-b}\right).\frac{c^2}{a^2}, \quad \text{as } a = c\cosh\alpha$$

$$= U^2\left(\frac{a+b}{a-b}\right).\frac{a^2-b^2}{a^2} \qquad \text{as } c = a^2 - b^2$$

Thus

$$(q_{max}) = \left[\frac{U(a+b)}{a}\right]$$

Example 6 :

A source is placed midway between two planes whose distance from one another is 2a. Find the equation of the streamlines when the motion is in two dimensions and show that those particles which at an infinite distance a/2 from one of the boundaries, issued from the source in a direction making an angle $\pi/4$ with it.

Solution :

The transformation $\zeta = ie^{\pi z/2a}$ transforms the strip of breadth 2a in the strip of breadth 2a in the z-plane into the upper half of the plane ζ-pane, the origin O′ in the z-pane being midway between the two walls. The points B, C coincide with (B_∞, C_∞), $\zeta = 0$.

When z = 0, $\zeta = i$, i.e., the point P in the ζ-plane.

Thus in the z-pane there is a source m at O′ and equal sink at infinite distance, so in the ζ-pane there will be a source m at P and a sink (−m) at (B, C) and hence an image souuce m at the point ζ = i.

Therefore,

$$w = -m\log(\zeta - i) - m\log(\zeta + i) + m\log\zeta$$

$$= -m\log\frac{\zeta^2+1}{\zeta} = -m\log(\zeta + \zeta^{-1})$$

$$= -m\log\left(ie^{\frac{\pi z}{2a}} - ie^{-\frac{\pi z}{2a}}\right) = -m\log i\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right)$$

$$-m\log\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right) - m\log i$$

Omitting the constant, we take

$$w = -m\log\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right)$$

or

$$w = -m\log(e^{cz} - e^{-cz}), \tag{1}$$

where

$$c = \pi/2a \tag{2}$$

so that

$$w = -m\log\left(e^{c(x+iy)} - e^{-c(x+iy)}\right).$$

Therefore

$$\phi + i\psi = -m\log[2\cos cy\sinh cx + 2i\sin cy\cosh cx]$$

and so

$$\psi = -m\tan^{-1}\frac{2\sin cy\cosh cx}{2\cos cy\sinh cx} = -m\tan^{-1}\left(\frac{\tan cy}{\tanh cx}\right).$$

Streamlines are given by ψ = constant, i.e., tan cy = K tanh cx,

$$\text{i.e.,}\quad \tan\frac{\pi y}{2a} = K\tanh\frac{\pi x}{2a} \quad [\text{using (2)}].$$

When $x = \infty$, $y = a/2$. Hence $K = 1$. Therefore streamlines become

$$\tan\frac{\pi y}{2a} = K\tanh\frac{\pi x}{2a}. \tag{3}$$

Diff. (3) w.r.t. x we have

$$\sec^2\frac{\pi y}{2a}\cdot\frac{dy}{dx} = \frac{1}{\cosh^2\frac{\pi x}{2a}}$$

i.e.,

$$\frac{dy}{dx} = \frac{\cos^2\frac{\pi y}{2a}}{\cosh^2\frac{\pi x}{2a}}.$$

Example 7 :

Use the transformation $\zeta = e^{\pi z/a}$ to find the streamlines of the motion in two dimensions due to a source midway between two infinite parallel boundaries (assume the liquid drawn off equally by sinks at the ends of the region). If the pressure tends to zero at the ends of the streams, prove that planes are pressed apart with a force which varies inversely as their distance from each other.

Solution :

We know that the transformation

$$\zeta = e^{\pi z/a} \tag{1}$$

transform the infinite strip A_∞, B_∞, C_∞, D_∞ in the z-plane with origin at O into the upper half in the ζ-plane with origin at (B, C) which coincide with B_∞, C_∞ at $\zeta = 0$. The point $z = ai/2$ goes to $\zeta = e^{\pi i/2} = i$ at the point P in ζ-plane. There is a source at O′ in the z-plane and equal sink at infinity, therefore in the ζ-plane there is a source of strength m at P, sink of strength $(-m)$ at (B, C) and an image source at $\zeta = -i$.

The complex potential is given by

$$\begin{aligned} w &= -m\log(\zeta - i) - m\log(\zeta + i) + m\log\zeta = -m\log(\zeta + \zeta^{-1}) \\ &= -m\log(e^{\pi z/a} + e^{-\pi z/a}), \quad \text{using (1)} \\ &= -m\log 2 - m\log\cosh(\pi z/a). \end{aligned}$$

Therefore

$$w = -m \log \cosh(\pi z / a), \quad \text{omitting the constant term in } \omega.$$

From (2),

$$q = \frac{dw}{dz} = -\frac{m\pi}{a} \tanh \frac{\pi z}{a}, \quad \text{and } q_{\infty} = \frac{m\pi}{a}.$$

We know that

$$\frac{p}{\rho} + \frac{1}{2} q^2 = \text{constant} = \frac{1}{2} q_{\infty}^2, \quad [p_{\infty} = 0]$$

i.e.,
$$\frac{p}{\rho} = \frac{\pi^2 m^2}{2a^2}\left(1 - \tanh^2 \frac{\pi z}{a}\right) = \frac{\pi^2 m^2}{a^2} \frac{1}{\cosh^2 \frac{\pi z}{a}} \tag{3}$$

Now, any point on the upper boundary is z = x + ia and hence (3) gives

$$\frac{p}{\rho} = \frac{\pi^2 m^2}{2a^2} \frac{1}{\cosh^2\left(\frac{\pi x}{a} + i\pi\right)} = \frac{\pi^2 m^2}{2a^2} \frac{1}{\cosh^2 \frac{\pi x}{a}}.$$

If F be the force with which the planes are pressed apart, then we have

$$F = 2\int_0^{\infty} p\,dx = \frac{\pi^2 \rho m^2}{a^2} \int_0^{\infty} \frac{1}{\cosh^2 \frac{\pi x}{a}} dx = \frac{\pi^2 \rho m^2}{a^2} \cdot \frac{a}{\pi}\left[\tanh \frac{\pi x}{a}\right]_0^{\infty} = \frac{\pi \rho m^2}{a},$$

showing that $F \propto \frac{1}{a}$ i.e. the force varies inversely as the distance between the planes apart.

1.23 Model Questions

Short Questions :

1. Show that the curves of equivelocity potential and stream lines intersect orthogonally.
2. Define stream function (or current function).
3. State the boundary conditions for the motion of a cylinder in a uniform stream.

4. Define flow and circulation for fluid motion.

5. Find the expression for the complex velocity potential in the case of motion of a fluid with circulation about a circular cylinder.

6. State Milne-Thomson Circle theorem, Blasius theorem and Kutta-Joukowski theorem.

7. What is meant by conformal mapping? When is it said to be isogonal?

8. Define Schwarz-Christoffel and Joukowski transformations.

9. What is meant by aerofoil? Define camber stating the assumptions required.

10. Define elliptic coordinates.

Broad Questions :

1. Discuss the motion of a circular (or/elliptic) cylinder moving in or infinite mass of the liquid at rest at infinity with velocity U in the direction of x-axis.

2. Discuss the motion of a liquid past a fixed circular (or elliptic) cylinder.

3. Show that if there is a streaming past a fixed circular (or elliptic) cylinder with velocity U in the negative direction of x-axis and there is a circulation of strength k, then the cylinder experiences an upward lift amounting ρkU, ρ being the density of the liquid.

4. Deduce the equation of motion of a circular cylinder moving in a liquid at rest at infinity. Hence show that the effect of the presence of the liquid is to reduce the extraneous force in the ratio $(\sigma - \rho) : (\sigma - \rho)$ where σ, ρ are the densities of the cylinder and liquid respectively.

5. Determine the velocity potential and the stream function at any point of a liquid contained between two coaxial circular cylinders.

6. State and prove Milne-Thomson circle theorem. Apply the theorem to find the complex potential of (i) a uniform flow with velocity U along negative x-axis past a fixed circular cylinder and (ii) a uniform stream at incidence β with positive x-axis.

7. State and prove Blasius theorem and the theorem of Kutta-Joukowski.

8. Determine the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity v in a direction making an angle β with the major axis of the cross-section of the cylinder.

9. Find the complex potential when an elliptic cylinder is rotating with constant angular velocity in an infinite mass of liquid at rest at infinity.

Problems :

1. Show that when a cylinder moves uniformly in a given straight line in an infinite liquid, the path of any point in the fluid is given by the equations

$$\frac{dz}{dt} = \frac{Va^2}{(z' - Vt)^2}; \frac{dz'}{dt} = \frac{Va^2}{(z - Vt)^2}.$$

where v = velocity of cylinder, a its radius, and z, z' are x + iy, x − iy and x, y are the coordinates measured from the starting point of the axis, along and perpendicular to its direction of motion.

2. If a long circular cylinder of radius a moves in a straight line at right angles to its length in liquid at rest at infinity, show that when a particle of liquid in the plane of symmetry, initially at distance b in advance of the axis of the cylinder has moved through a distance c, then the cylinder has moved through a distance

$$c + \frac{b^2 - a^2}{b + a\coth(c/a)}.$$

3. A circular cylinder of radius a and infinite length lies on a plane in an infinite depth of liquid. The velocity of liquid at a great distance from the cylinder is U perpendicular to the generators, and the motion is irrotational and two-dimensional. Verify that the stream function is the imaginary part of $w = \pi aU \coth(\pi a/z)$, where z is a complex variable, zero on the line of contact and real on the plane. Prove that the pressure at the two ends of the diameter of the cylinder normal to the plane differs by

$$(1/32)\pi^4\rho U^2.$$

4. The space between two infinitely long cylinders of radii a and b(a > b) respectively is filled with homogenous liquid of density ρ and is suddenly moved with velocity U perpendicular to the axis, the outer one is being kept

fixed. Show that the resultant impulsive pressure on a length l of the inner cylinder is

$$\pi\rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} U.$$

5. Prove that if 2a, 2b are axes of the cross-section of an elliptic cylinder placed across a stream in which the velocity at infinity is U parallel to the major axis of the cross-section, the velocity at a point (a cos η, b sin η) on the surface is

$$\frac{U(a+b)\sin\eta}{(b^2\cos^2\eta + a^2\sin^2\eta)^{1/2}}$$

and that, in consequence of the motion, the resultant thrust per unit length on that half of the cylinder on which the stream impinges is diminished by

$$\frac{2b^2\rho U^2}{a-b}\left[1-\left(\frac{a+b}{a-b}\right)^{1/2}\tan^{-1}A\left(\frac{a-b}{a+b}\right)^{1/2}\right],$$

where ρ is the density of the liquid.

6. An elliptic cylinder, the semi-axes of whose cross-sections are a and b, is moving with velocity U parallel to the major axis of the cross-section, through an infinite liquid of density ρ which is at rest at infinity, the pressure there being Π. Prove that in order that the pressure may everywhere be positive

$$\rho U^2 < \frac{2a^2\Pi}{2ab + b^2}.$$

7. An elliptic cylinder, semi-axes a and b, is held with its length perpendicular to, and its major axis making an angle θ with the direction of a stream of velocity V. Prove that the magnitude of the couple per unit length on the cylinder due to the fluid pressure is

$$\Pi\rho(a^2 - b^2)V^2\sin\theta\cos\theta$$

and determine its sense.

8. A rectangle open at infinity in the x-direction has solid boundaries along $x = 0$, $y = 0$ and $y = a$. Fluid of amount $2\pi m$ flows into and out of the rectangle at the corners $x = 0, y = 0$ and $x = 0, y = a$ respectively. Prove that the motion of the fluid is given by

$$\omega = 4m \log \tanh (\pi z/2a).$$

9. Show that the transformation $z = (a/\pi)\left\{\sqrt{(\zeta^2 - 1)} - \sec^{-1}\zeta\right\}, \zeta = e^{\pi\omega/aV}$ where $z = x + iy$, $0 = \phi + i\psi$, give the flow of a straight river of breadth a, running with velocity V at right angles to the straight shore of an otherwise unlimited sea of water into which it flows.

1.24 Summary

In this chapter, two-dimensional irrotational motion of an inviscid liquid past circular an elliptic cylinder has been considered. In addition, motion of these cylinders in the liquid has also been taken into account. Due to wide applications, Milne-Thomson circle theorem and Blasius theorem are discussed. Also a sketch of aerofoil is given.

Unit 2 □ Irrotational Motion in Three-dimensions

Structure

2.0 Introduction

We now describe irrotational motion in three dimensions with particular reference to the motion of a sphere, ellipsoid and solids of revolution in an infinite inviscid incompressible fluid. The stream function and velocity potential are obtained. It is to be noted that the powerful tool of the theory of complex functions cannot used in three dimensional problems.

2.1 Motion of a Sphere

We propose to study irrotational motion in three-dimensions with reference to the motion of a sphere. We shall consider spherical form of solution of the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0 \tag{1}$$

which, in spherical polar co-ordinates (r, θ, ω), reduces to

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{2}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta}+\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \omega^2}=0. \tag{2}$$

When there is symmetry about z-axis, ϕ is independent of ω and hence (2) reduces to

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{2}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta}=0. \tag{3}$$

Substituting $\phi = f(r)\cos\theta$ in (3), we see that

$$\left(\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}\right)\cos\theta-\frac{f(r)}{r^2}\cos\theta-\frac{\cos\theta}{r^2}f(r)=0,$$

so that f(r) satisfies

$$r^2\frac{d^2 f}{dr^2}+2r\frac{df}{dr}-2f(r)=0$$

which is a homogenous ordinary differential equation and the solution of the equation is of the form $f(r)=Ar+\frac{B}{r^2}$.

Hence the solution of the equation (3) can be taken as

$$\phi=f(r)\cos\theta=\left(Ar+\frac{B}{r^2}\right)\cos\theta. \tag{4}$$

2.1.1 Irrotational motion of liquid in which the sphere is moving :

Let a solid sphere of radius a is moving with velocity U through a homogeneous liquid which is at rest at infinity. Let O, the center of the sphere, be taken as the origin. We choose Oz in the direction of velocity U so that the motion is symmetrical about Oz. Let $P(r, \theta, \omega)$ be any point, and R' denote the region $r \geq a$ while R is the region $r \leq a$. $S(r = a)$ is the sphere which separates R and R'. If the motion is irrotational then the velocity can be expressed as $\vec{q} = -\vec{\nabla}\phi$, ϕ being the velocity potential. Thus the equation of continuity $\vec{\nabla} \cdot \vec{q} = 0$ gives

$$\nabla^2 \phi = 0 \qquad \text{in } R'$$

Since there is symmetry about the z-axis, ϕ is independent of ω and so $\nabla^2\phi = 0$ reduces to

$$\text{i.e. } \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} = 0. \quad \text{in } R'. \tag{5}$$

Boundary conditions are as follows :

(i) As the liquid is at rest at infinity, we must have

$$-\frac{\partial \phi}{\partial r} = 0 \quad \text{as } r \to \infty, \tag{6}$$

(ii) and as the normal velocity on the sphere is $U \cos\theta$, we must have

$$-\frac{\partial \phi}{\partial r} = U\cos\theta \quad \text{on } S(r = a). \tag{7}$$

Since ϕ is harmonic and normal derivative is prescribed at the boundary $S(r = a)$, so ϕ is unique except for an additive constant.

The boundary conditions (i) and (ii) suggest that ϕ must be of the form $f(r)\cos\theta$ and hence it is assumed as

$$\phi = \left(Ar + \frac{B}{r^2} \right)\cos\theta. \tag{8}$$

From (8)

$$-\frac{\partial \phi}{\partial r} = -\left(A - \frac{2B}{r^3} \right)\cos\theta. \tag{9}$$

Using (6) we get

$$A \cos\theta = 0 \quad \text{i.e.,} \quad A = 0. \tag{10}$$

Using (7) in (9) we get,

$$U\cos\theta = \frac{2B}{a^3}\cos\theta, \text{ for all values of } \theta,$$

so that

$$B = \frac{Ua^3}{2}. \tag{11}$$

Thus

$$\phi = \frac{Ua^3 \cos\theta}{2r^2} \tag{12}$$

which determines the velocity potential for the flow.

We now determine the equation of streamlines of the flow. The differential equation of the streamlines is

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\partial\phi/r\partial\theta}$$

i.e.,

$$\frac{dr}{\dfrac{Ua^3\cos\theta}{r^3}} = \frac{rd\theta}{\dfrac{Ua^3\sin\theta}{2r^3}}$$

so that

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta.$$

Integrating.

$$\log r = 2\log(\sin\theta) + \log C \quad (C \text{ is constant})$$

i.e.,

$$r = C\sin^2\theta$$

which is the equation of streamlines.

2.1.2 Equation of motion of a sphere :

We take the origin at the center of the sphere and the z-axis in the direction of motion. Let the sphere move with velocity U along the z-axis in an infinite mass of liquid at rest at infinity. The velocity potential of the motion is given by

$$\phi = \frac{Ua^3}{2r^2}\cos\theta$$

so that

$$\frac{\partial\phi}{\partial r} = -\frac{Ua^3}{r^3}\cos\theta.$$

Let P(a, θ, ω) be the spherical polar co-ordinates of any point on the surface of the sphere. Then the elementary area ds at P is adθ.a sin θdω. Again the value of $\phi\frac{\partial\phi}{\partial r}$ at P is given by

$$\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a} = -\frac{U^2 a\cos^2\theta}{2}. \tag{13}$$

The kinetic energy T_1 of the liquid is

$$T_1 = -\frac{\rho}{2}\iint \phi\frac{\partial\phi}{\partial n}\,ds,$$

integrated over the surface of the sphere, ρ being the density of the liquid. Using (13), we obtain

$$T_1 = -\frac{\rho}{2}\int_{\omega=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-\frac{U^2 a\cos^2\theta}{2}\right)\cdot(a^2\sin\theta d\theta d\omega)$$

$$= \frac{1}{4}U^2\rho a^3\left[\int_0^{\pi}\cos^2\theta\sin\theta d\theta\right]\times\left[\int_0^{2\pi}d\omega\right]$$

$$= \frac{\pi\rho a^3 U^3}{3} = \frac{1}{4}\cdot\frac{4}{3}\pi\rho a^3 .U^2 = \frac{M'U^2}{4} \tag{14}$$

where, $M' = \frac{4\pi a^3}{3}\rho$ is the mass of the liquid displaced by sphere.

Let σ be the density of the sphere and M be the mass of the sphere so that

$$M = \frac{4}{3}\pi\sigma a^3 \tag{15}$$

and K.E. of the sphere is $T_2 = \frac{1}{2}MU^2$. (16)

Let T be the total kinetic energy of the liquid and the sphere. Then

$$T = \frac{1}{2}\left(M + \frac{1}{2}M'\right)U^2, \text{ by (2) and (4).} \qquad (17)$$

Let Z be the external force parallel to the z-axis (i.e., in the direction of motion of sphere). Then from the principle of energy, we have

Rate of increase of total K.E. = rate at which work is being done

i.e., $$\frac{d}{dt}\left[\frac{1}{2}\left(M + \frac{1}{2}M'\right)U^2\right] = ZU$$

i.e., $$\left(M + \frac{1}{2}M'\right)U\dot{U} = ZU, \text{ where } \dot{U} = \frac{dU}{dt}$$

i.e., $$M\dot{U} = Z - \frac{1}{2}M'\dot{U}. \qquad (18)$$

Let Z' be the external force on the sphere when no liquid is present. Then from hydrostatical considerations, there exists a relation between Z and Z' of the form

$$Z = [(\sigma - \rho)/\sigma]Z' \qquad (19)$$

From (18) and (19), we have

$$M\dot{U} + \frac{1}{2}M'\dot{U} = [(\sigma - \rho)/\sigma]Z'$$

i.e., $$\left(M + \frac{1}{2}M'\right)\dot{U} = [(\sigma - \rho)/\sigma]Z'$$

i.e., $$M\dot{U} = \frac{M}{M + \frac{1}{2}M'}\frac{\sigma - \rho}{\sigma}Z'$$

i.e., $$M\dot{U} = \frac{\frac{4\pi\sigma a^3}{3}}{\frac{4\pi\sigma a^3}{3} + \frac{1}{2}\frac{4\pi\sigma a^3}{3}}\frac{\sigma - \rho}{\sigma}Z'$$

i.e., $$M\dot{U} = \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho}Z'. \qquad (20)$$

Equation (20) shows that the whole effect of the presence of the liquid is to reduce the external force in the ratio $(\sigma - \rho) : \left(\sigma + \frac{1}{2}\rho\right)$.

2.1.3 Fixed sphere in a uniform stream :

Let there be a uniform stream of velocity V in the negative direction of z-axis and the sphere be kept fixed. $R'(r \geq a)$ and $R(r \leq a)$ are the two regions separated by the sphere $S(r = a)$. The motion is irrotational and the velocity potential satisfies

$$\nabla^2\phi = 0 \quad \text{in } R'. \tag{21}$$

Boundary conditions are as follows :

(i) As the sphere is fixed, we have

$$\frac{\partial\phi}{\partial r} = 0, \quad \text{on} \quad S(r = a) \tag{22}$$

(ii) the infinity condition gives

$$\phi \sim Vz \quad \text{as} \quad r \to \infty. \tag{23}$$

The boundary condition (ii) suggests

$$\phi = Vz + \phi_1 \tag{24}$$

where $\phi_1 \to 0$ as $r \to \infty$.

Equation (24) gives

$$\nabla^2\phi_1 = 0 \quad \text{in } R' \tag{25}$$

and from (24) by using (2) we get

$$\frac{\partial\phi_1}{\partial r} = V\frac{\partial z}{\partial r} = -V\cos\theta \quad \text{on } S. \tag{26}$$

The conditions (25) and (26) suggest that ϕ_1 must be of the form

$$\phi_1 = \left(Ar + \frac{B}{r^2} \right)\cos\theta \tag{27}$$

A, B being constants.

Using the conditions (25) and (26) we get,

$$\phi_1 = \frac{a^3 V}{2r^2}\cos\theta$$

Hence

$$\phi = Vr\cos\theta + \frac{1}{2}\frac{a^3 V}{r^2}\cos\theta.$$

Here $Vr\cos\theta$ is the velocity potential due to the uniform stream and $\frac{a^3V}{2r^2}\cos\theta$ is the velocity potential due to the presence of sphere.

Now we determine the lines of flow relative to the sphere.

The streamlines are given by the differential equation

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\partial\phi/r\partial\theta}$$

i.e.,
$$\frac{dr}{V\left(1-\frac{a^3}{r^3}\right)\cos\theta} = \frac{rd\theta}{-V\left(1+\frac{a^3}{2r^3}\right)\sin\theta}$$

i.e.,
$$-2\cot\theta d\theta = \frac{2r^3+a^3}{r^3-a^3}\cdot\frac{dr}{r} = \left(\frac{3r^2}{r^3-a^3}-\frac{1}{r}\right)dr.$$

Integrating

$$-2\log\sin\theta = \log(r^3-a^3) - \log r - \log c$$

where log c is integration constant.

i.e.,
$$r^2\sin^2\theta\left(1-\frac{a^3}{r^3}\right) = c.$$

On the surface of the sphere

$$q_r = \left(-\frac{\partial\phi}{\partial r}\right)_{r=a} = 0$$

$$q_\theta = \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=a} = \frac{3V\sin\theta}{2}.$$

We note that $q_\theta = 0$ for $\theta = 0, \pi$ and it is minimum for $\theta = \pi/2, 3\pi/2$ and the minimum value is $\frac{3V}{2}$.

Hence $\theta = 0, \pi$ are the stagnation points on $r = a$.

2.1.4 Moving concentric spheres :

Let the region between two concentric spheres of radii a and b(> a) be filled with liquid which is homogenous and incompressible, R be the region between two concentric spheres i.e., $R(a < r < b)$. Impulses $\vec{I}_1$ and $\vec{I}_2$ are applied on the spheres $S_1(r = a)$ and $S_2(r = b)$ respectively in the z direction so that the two spheres start to move with velocities U and V respectively in the positive direction of z-axis. We intend to determine the resulting motion.

Since the motion is irrotational and symmetric about z-direction, the velocity potential ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} = 0. \quad \text{in } R : (a < r < b). \tag{28}$$

The boundary conditions are

$$(i) \quad -\frac{\partial \phi}{\partial r} = U \cos\theta \quad \text{on} \quad S_1(r = a), \tag{29}$$

$$(ii) \quad -\frac{\partial \phi}{\partial r} = V \cos\theta \quad \text{on} \quad S_2(r = b). \tag{30}$$

The boundary conditions (i) and (ii) suggest that ϕ must be of the form

$$\phi = \left(Ar + \frac{B}{r^2} \right)\cos\theta \tag{31}$$

where A and B are constants.

From (31) we get,

$$-\frac{\partial \phi}{\partial r} = -\left(A - \frac{2B}{r^3} \right)\cos\theta. \tag{32}$$

Using (29) and (30) in (32) we get

$$A = \frac{Ua^3 - Vb^3}{b^3 - a^3} \text{ and } B = \frac{(U - V)a^3 b^3}{2(b^3 - a^3)}$$

Therefore, for the starting motion, the velocity potential is given by

$$\phi = \frac{1}{b^3 - a^3}\left[(a^3 U - b^3 V) r + \frac{a^3 b^3 (U - V)}{2r^2} \right]\cos\theta. \tag{33}$$

In this case, the impulsive pressures on the boundaries when the motion is started from rest, are $\rho\phi$ so that these are given by

$$\varpi_1 = \frac{a\cos\theta}{b^3 - a^3}\left[\left(a^3 + \frac{b^3}{2} \right) U - \frac{3b^3}{2} V \right]\rho \quad \text{on } S_1$$

and

$$\varpi_2 = \frac{b\cos\theta}{b^3 - a^3}\left[\frac{3a^3}{2}U - \left(a^3 + \frac{b^3}{2}\right)V\right]\rho \quad \text{on } S_2.$$

The impulsive thrust on the inner boundary is therefore,

$$I_1 = \int_0^{\pi} \varpi_1 \cos\theta . 2\pi a^2 \sin\theta d\theta$$

$$= \frac{4\pi a^3 \rho}{3}\left[\left(a^3 + \frac{b^3}{2}\right)U - \frac{3b^3}{2}V\right] / (b^3 - a^3).$$

Similarly, on the outer boundary the impulsive thrust is

$$I_2 = \frac{4\pi b^3 \rho}{3}\left[\frac{3a^2}{2}U - \left(\frac{a^3}{2} + b^3\right)V\right] / (b^3 - a^3).$$

2.2 Axi-symmetric Motion

A motion is called axi-symmetric if it is symmetric about a line, called the axis. Here the motion is the same in every plane through the axis and the plane is called the meridian plane. Now taking the axis of symmetry as z-axis and using the cylindrical coordinate system, every field variable is a function of $\varpi(= (y^2 + x^2)^{1/2})$ and z only.

2.2.1 Stokes' stream function :

Let the axis of symmetry be the axis of z and let $\varpi(= (y^2 + x^2)^{1/2})$ denote distance from the axis. Let u, v denote the components of velocity in the direction of the z and ϖ. Then the equation of continuity is obtained by equating to zero the flow out of the annular space obtained by revolving a small rectangle $d\varpi dz$ around the axis. The total flow out parallel to z is $\frac{\partial}{\partial z}(u 2\pi\varpi d\varpi)dz$ and parallel to ϖ, the total flow out is $\frac{\partial}{\partial \omega}(v.2\pi\varpi dz)d\omega$, so that by equating the sum to zero we get the equation of continuity as

$$\frac{\partial}{\partial z}(u\varpi) + \frac{\partial}{\partial \varpi}(v\varpi) = 0.$$

This is, however, the condition that $v\varpi dz - u\varpi d\varpi$ may be an exact differential, and if we denote this by $d\psi$, we get

$$u = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}, \; v = \frac{1}{\varpi}\frac{\partial \psi}{\partial z}$$

This function ψ is called Stokes' stream function.

The streamlines are given by

$$\frac{dz}{u} = \frac{d\varpi}{v}$$

$$\text{i.e., } \varpi(v dz - u d\varpi) = 0,$$

that is, by $d\psi = 0$. Hence the equation ψ = constant represents stream lines.

A property of Stokes' stream function is that 2π times the difference of its values at two points in the same meridian plane is equal to the flow across the annular surface obtained by the revolution round the axis joining the points. For, if ds be an element of the curve and θ its inclination to its axis, the flow outwards across the surface of revolution is

$$\int (v\cos\theta - u\sin\theta).2\pi\varpi ds = 2\pi\int\left(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial \varpi}d\varpi\right) = 2\pi\int d\psi = 2\pi(\psi_2 - \psi_1).$$

We might also define the value of Stokes' stream function at any point P as $\frac{1}{2\pi}$ of the amount of flow across a surface obtained by revolving a curve AP round the axis, A being a fixed point in the meridian plane through P; for, this makes

$$\psi = \frac{1}{2\pi}\int_A^P (v\cos\theta - u\sin\theta).2\pi\varpi ds$$

$$= \int_A^P (v\varpi dz - u\varpi d\varpi)$$

and by varying the position of P, we get as before,

$$u = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} \text{ and } v = \frac{1}{\varpi}\frac{\partial \psi}{\partial z}. \tag{34}$$

2.2.2 Irrotational axi-symmetric motion :

Let us consider an irrotational motion for which the velocity potential is ϕ. Therefore,

$$u = -\frac{\partial \phi}{\partial z}, \; v = -\frac{\partial \phi}{\partial \varpi}. \tag{35}$$

Again Stokes' stream function always exists such that

$$u = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} \text{ and } v = \frac{1}{\varpi}\frac{\partial \psi}{\partial z}. \tag{36}$$

Thus

$$\frac{\partial \phi}{\partial z} = \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}, \ \frac{\partial \phi}{\partial \varpi} = -\frac{1}{\varpi}\frac{\partial \psi}{\partial z} \tag{37}$$

From (37)

$$\frac{\partial}{\partial \varpi}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right)$$

$$\text{and } \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial \varpi}\right) = -\frac{\partial}{\partial z}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial z}\right)$$

so that

$$\frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right) = -\frac{\partial}{\partial z}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial z}\right) \qquad \left[\because \frac{\partial^2 \phi}{\partial \varpi \partial x} = \frac{\partial^2 \phi}{\partial x \partial \varpi}\right]$$

$$\text{i.e.,} \quad -\frac{1}{\varpi^2}\frac{\partial \psi}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial^2 \psi}{\partial \varpi^2} = -\frac{1}{\varpi}\frac{\partial^2 \psi}{\partial z^2},$$

$$\text{i.e.,} \quad \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} = 0. \tag{38}$$

Again from (37)

$$\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial \varpi}\right) = \frac{\partial}{\partial z}\left(\varpi\frac{\partial \phi}{\partial z}\right).$$

and

$$\frac{\partial}{\partial \varpi}\left(\frac{\partial \psi}{\partial z}\right) = -\frac{\partial}{\partial \varpi}\left(\varpi\frac{\partial \phi}{\partial \varpi}\right)$$

so that

$$\frac{\partial}{\partial z}\left(\varpi\frac{\partial \phi}{\partial z}\right) = -\frac{\partial}{\partial \varpi}\left(\varpi\frac{\partial \phi}{\partial \varpi}\right) \qquad \left[\because \frac{\partial^2 \psi}{\partial z \partial \varpi} = \frac{\partial^2 \psi}{\partial \varpi \partial z}\right]$$

$$\text{i.e.,}\quad \varpi\frac{\partial^2\phi}{\partial z^2} = -\varpi\frac{\partial^2\phi}{\partial\varpi^2} - \frac{\partial\phi}{\partial\varpi},$$

$$\text{i.e.,}\quad \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial\phi}{\partial\varpi} = 0. \tag{39}$$

Equations (38) and (39) show that ϕ and ψ are not interchangeable in the way that is applied to the velocity potential and stream function of two-dimensional irrotational motion.

Now we rewrite (38) and (39) in polar co-ordinates. Let q_r and q_θ be the velocities in the directions of dr and rdθ. Then, since $\varpi = r\sin\theta$ and the velocity from right to left across ds is $\frac{1}{\varpi}\frac{\partial\psi}{\partial s}$, we get

$$q_r = -\frac{1}{\varpi}\frac{\partial\psi}{r\partial\theta} = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},$$

$$q_\theta = \frac{1}{\varpi}\frac{\partial\psi}{\partial r} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{40}$$

But in irrotational motion, we know that

$$q_r = -\frac{\partial\phi}{\partial r},\ q_\theta = -\frac{\partial\phi}{r\partial\theta} \tag{41}$$

and since $$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} \text{ and } \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \tag{42}$$

so $$\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \frac{\partial^2\phi}{\partial\theta\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)$$

$$\text{i.e.,}\quad r^2\frac{\partial^2\psi}{\partial r^2} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0. \tag{43}$$

Let $\mu = \cos\theta$ so that

$$\sin\theta\frac{\partial}{\partial\mu} = -\frac{\partial}{\partial\theta}, \tag{44}$$

then (43) reduces to

$$r^2 \frac{\partial^2 \psi}{\partial r^2} \sin^2 \theta \frac{\partial}{\partial \mu}\left(\frac{\partial \psi}{\partial \mu}\right) = 0. \tag{45}$$

Similarly eliminating ψ from (42), we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

$$\text{i.e.,} \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right] = 0 \tag{46}$$

which is Laplace's equation and has solution of the forms $r^n P_n(\mu)$ and $r^{-n-1}P_n(\mu)$, $P_n(\mu)$ being the Legendre polynomial of degree n.

Again from (42), we have

$$\frac{\partial \psi}{\partial \mu} = -r^2 \frac{\partial \phi}{\partial r} = -nr^{n+1} P_n \quad \text{or} \quad (n+1) r^{-n} P_n, \tag{47}$$

$$\frac{\partial \psi}{\partial r} = (1-\mu^2)\frac{\partial \phi}{\partial \mu} = (1-\mu^2) r^n \frac{\partial P_n}{\partial \mu} \quad \text{or} \quad (1-\mu^2) r^{-n-1} \frac{\partial P_n}{\partial \mu}. \tag{48}$$

On integration, (48) gives us possible solutions for ψ as

$$\psi = \frac{(1-\mu^2)}{n+1} r^{n+1} \frac{\partial P_n}{\partial \mu} \quad \text{or} \quad -\frac{(1-\mu^2)}{n}\frac{1}{r^n}\frac{\partial P_n}{\partial \mu}. \tag{49}$$

2.2.3 Solids of revolution moving along their axes in an infinite mass of liquid :

Suppose that a solid moves along Ox with velocity U and let Ox be the axis of revolution. Since the motion is symmetrical about Ox, Stokes' stream function exists.

Now the normal velocity of the liquid in contact with the surface at P is $-\frac{1}{\varpi}\frac{\partial \psi}{\partial s}$. On the boundary, we have

$$-\frac{1}{\varpi}\frac{\partial \psi}{\partial s} = \text{velocity of the solid along normal}$$

$$\text{i.e., } -\frac{1}{\varpi}\frac{\partial \psi}{\partial s} = U\cos\theta = U\frac{\partial \varpi}{\partial s}, \text{ where } \cos\theta = \frac{\partial \varpi}{\partial s}$$

$$\text{i.e., } d\psi = -U\varpi d\varpi$$

Integrating,

$$\psi = -\frac{U\varpi^2}{2} + \text{constant}$$

$$\text{i.e., } \psi = -\frac{Ur^2 \sin^2\theta}{2} + \text{constant, where } \varpi = r\sin\theta \tag{50}$$

$$\text{i.e., } \psi = -\frac{U(1-\mu^2)}{2} + \text{constant, where } \mu = \cos\theta \tag{51}$$

which is the boundary condition at P.

Again ψ must satisfy the equation

$$r^2\frac{\partial^2\psi}{\partial r^2} + (1-\mu^2)\frac{\partial^2\psi}{\partial \mu^2} = 0, \text{ where } \mu = \cos\theta \tag{52}$$

and it is known that (52) has solutions of the form $\frac{1-\mu^2}{n+1} r^{n+1}\frac{\partial P_n}{\partial \mu}$ and $\frac{1-\mu^2}{nr^n}\frac{\partial P_n}{\partial \mu}$

As an example, we consider the case of a sphere of radius a. Then with r = a in (51), we must have

$$\psi = -\frac{Ua^2}{2}(1-\mu^2) + C \tag{53}$$

Taking n = 1 in (49), we have the solution of the form

$$\psi = A\frac{1-\mu^2}{r}, \tag{54}$$

then at the boundary we must have

$$\frac{A(1-\mu^2)}{a} = -\frac{Ua^2}{2}(1-\mu^2) + C$$

for all values of μ. This requires that C = 0 and $A = -\frac{Ua^3}{2}$. Hence putting these values and noting that $\mu = \cos\theta$, (54) gives

$$\psi = -\frac{Ua^3\sin^2\theta}{2r} \tag{55}$$

Again we know that

$$(1-\mu^2)\frac{\partial\phi}{\partial\mu} = \frac{\partial\psi}{\partial r} = \frac{Ua^3 \sin^2\theta}{2r^2}$$

$$\text{i.e.,}\ \frac{\partial\phi}{\partial\mu} = \frac{Ua^3}{2r^2}.$$

Integrating

$$\phi = \frac{Ua^3}{2r^2}\mu = \frac{Ua^3}{2r^2}\cos\theta. \tag{56}$$

2.3 Ellipsoidal Coordinate System

Let us consider the equation

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1,\ a > b > c \tag{57}$$

where θ is a parameter. This represents a family of confocal central conicoids. The above equation can be reduced to a cubic equation of θ, given by

$$F(\theta) \equiv x^2(b^2+\theta)(c^2+\theta) + y^2(a^2+\theta)(c^2+\theta) + z^2(a^2+\theta)(b^2+\theta)$$
$$-(a^2+\theta)(b^2+\theta)(c^2+\theta) = 0. \tag{58}$$

Now

$F(-\infty) = +\text{ve},\ F(-a^2) = +\text{ve},\ F(-b^2) = -\text{ve},\ F(-c^2) = +\text{ve},\ F(\infty) = -\text{ve}.$

Hence we conclude that $F(\theta)$ has three real roots λ, μ, ν such that

$$-a^2 < \nu < -b^2 < \mu < -c^2 < \lambda.$$

Thus through any fixed point (x, y, z), there are three conicoids represented by

λ = constant, μ = constant, ν = constant

It may be noted that

λ = constant represents an ellipsoid,

μ = constant represents a hyperboloid of one sheet,

and

ν = constant represents a hyperboloid of two sheets.

Now we write

$$f(\lambda) \equiv \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0. \tag{58}$$

Differentiating with respect to x we get,

$$f'(\lambda)\frac{\partial\lambda}{\partial x} = \frac{2x}{a^2+\lambda}$$

$$\text{i.e., } \frac{\partial\lambda}{\partial x} = \frac{1}{f'(\lambda)}\frac{2x}{a^2+\lambda}.$$

Similarly

$$\frac{\partial\lambda}{\partial y} = \frac{1}{f'(\lambda)}\frac{2y}{b^2+\lambda},$$

$$\frac{\partial\lambda}{\partial z} = \frac{1}{f'(\lambda)}\frac{2z}{c^2+\lambda}.$$

Direction cosines of the normal to the surface λ = constant are proportional to $\left(\frac{\partial\lambda}{\partial x}, \frac{\partial\lambda}{\partial y}, \frac{\partial\lambda}{\partial z}\right)$. Similarly, direction cosines of the normal to the surface μ = constant are proportional to $\left(\frac{\partial\mu}{\partial x}, \frac{\partial\mu}{\partial y}, \frac{\partial\mu}{\partial z}\right)$. Now the cosine of the angle between these normals is proportional to

$$\frac{\partial\lambda}{\partial x}\frac{\partial\mu}{\partial x} + \frac{\partial\lambda}{\partial y}\frac{\partial\mu}{\partial y} + \frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial z}$$

$$= \frac{1}{f'(\lambda)f'(\mu)}\left[\frac{4x^2}{(a^2+\lambda)(a^2+\mu)} + \frac{4y^2}{(b^2+\lambda)(b^2+\mu)} + z\right]$$

$$= \frac{4}{f'(\lambda)f'(\mu)}(f(\lambda) - f(\mu)) \tag{59}$$

which vanishes if $f(\lambda) = 0$, $f(\mu) = 0$. Hence λ, μ, ν give the system of orthogonal curvilinear coordinates called ellipsoidal co-ordinates. Again λ, μ, ν are the roots of $F(\theta) = 0$, so that $F(\theta)$ can be written as

$$F(\theta) = (\lambda - \theta)(\mu - \theta)(\nu - \theta).$$

Let us put $\theta = -a^2, -b^2, -c^2$ in (35) successively and we get

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-a^2)(b^2-c^2)}$$

$$z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}$$

Now if ds is an element then

$$ds^2 = h_1^2 d\lambda^2 + h_2^2 d\mu^2 + h_3^2 d\nu^2$$

where

$$h_1^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2,$$

$$h_2^2 = \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2,$$

$$h_3^2 = \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2.$$

Now it is easy to see that

$$4h_1^2 = \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2};$$

similarly,

$$4h_2^2 = \frac{x^2}{(a^2+\mu)^2} + \frac{y^2}{(b^2+\mu)^2} + \frac{z^2}{(c^2+\mu)^2},$$

$$4h_3^2 = \frac{x^2}{(a^2+\nu)^2} + \frac{y^2}{(b^2+\nu)^2} + \frac{z^2}{(c^2+\nu)^2}.$$

We can write

$$f(\theta) = \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)},$$

then

$$-f'(\lambda) = \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

Thus

$$4h_1^2 = \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)},$$

$$4h_2^2 = \frac{(\mu-\lambda)(\mu-\nu)}{(a^2+\mu)(b^2+\mu)(c^2+\mu)},$$

$$4h_3^2 = \frac{(\nu-\lambda)(\nu-\mu)}{(a^2+\nu)(b^2+\nu)(c^2+\nu)}.$$

So the Laplace operator in ellipsoidal coordinates is

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\lambda}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial\lambda}\right)+\frac{\partial}{\partial\mu}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial\mu}\right)+\frac{\partial}{\partial\nu}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial\nu}\right)\right]$$

$$=(\mu-\nu)\left(K_\lambda\frac{\partial}{\partial\lambda}\right)^2\phi+(\nu-\lambda)\left(K_\mu\frac{\partial}{\partial\mu}\right)^2\phi+(\lambda-\mu)\left(K_\nu\frac{\partial}{\partial\nu}\right)^2\phi$$

where

$$K_\lambda = (a^2-\lambda)(b^2-\lambda)(c^2-\lambda),$$

$$K_\mu = (a^2+\mu)(b^2+\mu)(c^2+\mu),$$

$$K_\nu = (a^2+\nu)(b^2+\nu)(c^2+\nu),$$

Solutions of this Laplace equation are called ellipsoidal harmonics.

2.3.1 Translatory motion of an ellipsoid :

We consider the ellipsoid S : $\lambda = 0$,

$$\text{i.e., } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0. \tag{60}$$

which moves through a liquid in the direction of x-axis with velocity U. Since the motion is irrotational, the velocity potential ϕ satisfies

$$\nabla^2\phi = 0 \quad \text{for} \quad \lambda \geq 0.$$

The boundary conditions are

(i) $-\frac{\partial\phi}{\partial n} = U\cos\theta_x$, on $\lambda = 0$

where θ_x is the angle between the normal and x-axis,

$$\text{i.e., } -\frac{\partial\phi}{\partial\lambda} = -U\frac{\partial x}{\partial\lambda},\ \lambda = 0,$$

since $dn = h_1\, d\lambda$, $\cos\theta_x = \frac{1}{h_1}\frac{\partial x}{\partial\lambda}$. Thus

$$\phi = -Ux \quad \text{on} \quad \lambda = 0. \tag{61}$$

(ii) ϕ is regular at infinity

$$\text{i.e., } \phi \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{62}$$

For solution of the Laplace equation (60) in the ellipsoidal coordinate system, we take

$$\phi = Cx\int_{\lambda}^{\infty}\frac{dt}{(a^2+t)K_1} \text{ which tends to 0 as } \lambda \to \infty \tag{63}$$

where C is constant.

Using the boundary condition (61) in (62) we get

$$-U\frac{\partial x}{\partial\lambda} = C\frac{\partial x}{\partial\lambda}\int_0^{\infty}\frac{dt}{(a^2+t)K_1} - \frac{Cx}{a^2 .abc}, \quad \text{where } \lambda = 0$$

Again

$$\frac{\partial x}{\partial\lambda} = \frac{x}{2a^2}, \quad \text{when } \lambda = 0$$

therefore,

$$C = \frac{abcU}{2-\alpha_0} \text{ where } \alpha_0 = abc\int_0^{\infty}\frac{dt}{(a^2+t)K_1}. \tag{64}$$

Thus finally we get

$$\phi = \frac{abcUx}{2-\alpha_0}\int_{\lambda}^{\infty}\frac{dt}{(a^2+t)^{3/2}(b^2+t)^{1/2}(a^2+t)^{1/2}} \tag{65}$$

and on the ellipsoid we have from (64)

$$\phi = \frac{\alpha_0 xU}{2-\alpha_0}. \tag{66}$$

The kinetic energy of the liquid is

$$T = -\frac{1}{2}\rho\int\phi\frac{\partial\phi}{\partial n}ds = \frac{\alpha_0\rho U^2}{2(2-\alpha_0)}\int x\cos\theta_x\, ds.$$

Since $\cos\theta_x\, ds$ is the projection on the plane $x = 0$ of the area ds of the surface, and the last integral gives the volume of the ellipsoid as $\frac{4\pi abc}{3}$ we find

$$T = \frac{M'\alpha_0 U^2}{2(2-\alpha_0)}$$

where M′ is the mass of liquid displaced by ellipsoid.

When the ellipsoid has, in addition, velocity components V, W parallel to y-axis and z-axis, we get, by superposing the results analogous to (66), the velocity potential to be

$$\frac{abcU_x}{2-\alpha_0}\int_\lambda^\infty \frac{dt}{(a^2+t)K_t} + \frac{abcV_y}{2-\beta_0}\int_\mu^\infty \frac{dt}{(b^2+t)K_t} + \frac{abcW_z}{2-\gamma_0}\int_\nu^\infty \frac{dt}{(c^2+t)K_t}$$

where β_0, γ_0 are defined by writing $b^2 + t$, $c^2 + t$ for $a^2 + t$ in (5).

2.4 Source, Sink, Doublet

Source :

Source is a point at which liquid is created and distributed at a uniform rate and the liquid flows outward symmetrically in all directions from the point. If the rate of emission of the volume of liquid is $4\pi m$, then m is called the *strength of the source*. When the rate of emission is constant then the source is called steady.

Let us consider a steady irrotational motion due to the source of strength m. The volume of the liquid flowing out in a spherical surface of radius r and the source at its center must be equal to the volume of liquid created per unit time. Let ϕ be the velocity potential due to a simple source of strength m, and the liquid be at rest at infinity. Then

$$4\pi m = \text{flux of liquid across the spherical surface} = -\frac{\partial\phi}{\partial r}\cdot 4\pi r^2$$

So,

$$\phi = \frac{m}{r} + \text{constant}$$

Since constant velocity potential does not change the motion, we may neglect the constant or may redefine the velocity potential by including the constant in it.

Sink :

A *sink* is a source of negative strength.

Note : A source or sink implies creation or annihilation of fluid at a point. Both are points at which the velocity potential is infinite. A source and sink are purely abstract conception but they are to be considered due to exigencies of analysis.

Doublet :

A combination of source and sink of equal strength m at a small distance δs apart, when the limit of m is infinitely large and δs is infinitely small, but $m\delta s$ remains finite and equal to μ, then it is called a *doublet* of strength μ and the line δs taken from – m to m is called the axis of the doublet. Let $\vec{\nu}$ denotes the direction of the axis of the doublet. So,

$$[\phi]_P = \lim_{m\delta s \to \mu}\left[-m\left(\frac{1}{r}\right)_Q + m\left(\frac{1}{r}\right)_{Q'}\right] = \lim_{m\delta s \to \mu}\left[m\delta s \frac{\left(\frac{1}{r}\right)_{Q'} - \left(\frac{1}{r}\right)_Q}{\delta s}\right] = \mu \frac{\partial}{\partial \nu}\left(\frac{1}{r}\right)$$

where the source is at Q and the sink is at Q′, and in the limit both Q and Q′ tend to P. Thus

$$[\phi]_P = \mu \frac{\partial}{\partial \nu}\left(\frac{1}{r}\right) = -\frac{\mu}{r^2}\frac{\partial r}{\partial \nu}.$$

Again, since $r = -\nu\cos\theta$

$$[\phi]_P = -\frac{\mu}{r^2}\frac{\partial}{\partial \nu}(-\nu\cos\theta) = \frac{\mu\cos\theta}{r^2}$$

2.5 Images

If in a liquid a surface S can be drawn across which there is no flow, then any systems of sources, sinks and doublets on opposite sides of this surface may be said to be images of one another with regard to the surface. And if the surface S be regarded as a rigid boundary and the liquid is removed from one side of it, the motion on the other side will remain unaltered.

2.5.1 Image of a source with respect to a rigid plane :

Let S(x = 0) be a fixed plane and a source of strength m be placed at Q(a, 0, 0) in front of S (see figure 1.1.). Let Q'(– a, 0, 0) be another point which is image point of Q with respect to S. Let P be any fixed point and r_1, r_2 be the distances of Q, Q' respectively from P.

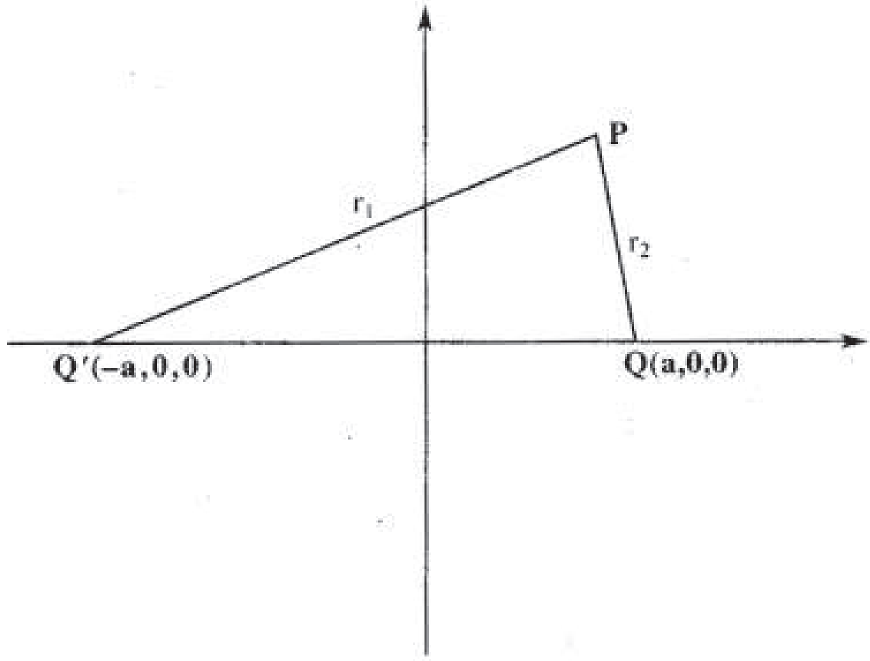

Figure 2.1

Since the motion is irrotational on the right of S(x = 0) due to the source at Q, so

$$\nabla^2\phi = 0 \text{ in } R : x \geq 0 \text{ except at } Q,$$

therefore,

$$\phi \sim \frac{m}{r_1} \text{ near } Q$$

where r_1 is the distance from Q ($r_1 \to 0$), and also ϕ is regular at infinity. Again, $\frac{\partial \phi}{\partial x} = 0$ on $S(x = 0)$.

Now we set

$$\phi = \frac{m}{r_1} + \phi_1$$

where ϕ_1 is due to the presence of the rigid wall. Then

$$\nabla^2 \phi_1 = \nabla^2 \phi - \nabla^2 \left(\frac{m}{r_1} \right) = 0$$

and

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi}{\partial x} - m \frac{\partial}{\partial x} \left(\frac{1}{r_1} \right) = -m \frac{\partial}{\partial x} \left(\frac{1}{r_1} \right) \text{ on S.}$$

Now

$$r_1^2 = (x - a)^2 + y^2 + z^2, \; r_2^2 = (x + a)^2 + y^2 + z^2$$

$$\frac{\partial \phi_1}{\partial x} = -m \frac{\partial}{\partial x} \left(\frac{1}{r_1} \right) = \frac{ma}{r_1^3} \text{ on S}$$

We choose $\phi_1 = \frac{m}{r_2}$, the reason for this is as follows :

$$\frac{\partial \phi_1}{\partial x} = -m \frac{\partial}{\partial x} \left(\frac{1}{r_2} \right) = \frac{ma}{r_2^3} \text{ on } x = 0.$$

Therefore, on $x = 0$

$$\frac{ma}{r_1^3} = \frac{ma}{r_2^3}, \text{ which is obvious.}$$

Hence

$$\phi_1 = \frac{m}{r_2} \text{ in R.}$$

Therefore,

$$\phi = \frac{m}{r_1} + \frac{m}{r_2}$$

This shows that the image of a point source with respect to a point is a point source of same strength at the image point.

2.5.2 Image of a source in front of a sphere :

Let S(r = a) be a fixed sphere of radius a and a source of strength m be placed on z-axis at a distance f from the center of the sphere. R(r ≤ a), R′(r ≥ a) are two regions separated by the sphere S(r = a) (See figure 2.2).

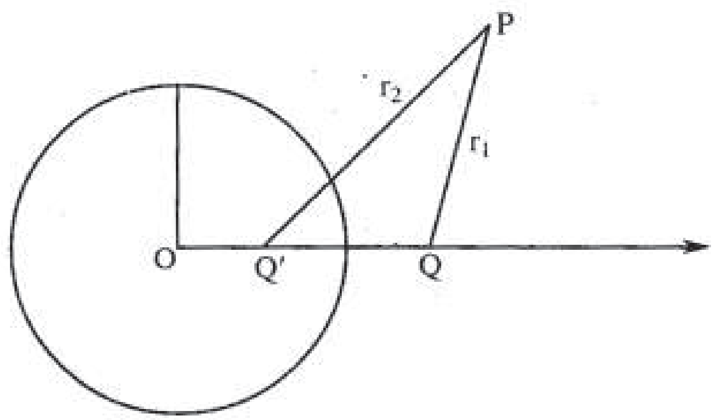

Figure 2.2

Let Q′ be the inverse point of Q with respect to the sphere then, $OQ' = \frac{a^2}{f}$.

Let P be any field point, which is at a distance r_1 and r_2 from Q and Q′ respectively and (r, θ, ω) be the co-ordinates of P.

The velocity potential ϕ is composed of two parts, one is ϕ_1 which is due to the source of strength m and another is ϕ_2 which is due to the presence of spherical boundary. The later part will be the velocity potential of the required system.

As the motion is irrotational, the velocity potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0 \text{ in } R' \text{ except at } Q.$$

and the conditions

(i) $\phi \sim \dfrac{m}{r_1}$ near Q where r_1 is the distance from Q,

(ii) ϕ is regular at infinity

(iii) and since S is fixed, $\dfrac{\partial\phi}{\partial r} = 0$. on S.

Let us set

$$\phi = \frac{m}{r_1} + \phi_1$$

then

$$\nabla^2\phi_1 = 0 \text{ and } \frac{\partial\phi_1}{\partial r} = \frac{\partial\phi}{\partial r} - m\frac{\partial}{\partial r}\left(\frac{1}{r_1}\right) = -m\frac{\partial}{\partial r}\left(\frac{1}{r_1}\right) \text{ on S.}$$

Now,

$$\frac{1}{r_1} = \frac{1}{\sqrt{r^2 + f^2 - 2rf\cos\theta}} = \frac{1}{f}\frac{1}{\sqrt{\left(\frac{r}{f}\right)^2 - \frac{2f}{r}\cos\theta + 1}} = \frac{1}{f}\sum\frac{r^n}{f^n}P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is Legendre's polynomial.

Again, $r > OQ' = \dfrac{a^2}{f} = b < a$

$$\frac{1}{r_2} = \frac{1}{\sqrt{r^2 + b^2 - 2rb\cos\theta}} = \frac{1}{f}\sum_{n=0}^{\infty}\frac{b^n}{r^{n+1}}P_n(\cos\theta).$$

On S,

$$\frac{\partial\phi_1}{\partial r} = -m\sum_{n=0}^{\infty}\frac{na^{n-1}}{f^{n+1}}P_n(\cos\theta).$$

Let us take

$$\phi_1 = \sum_{n=0}^{\infty}\frac{A_n}{r^{n+1}}P_n(\cos\theta)$$

so that   is regular at infinity, and on S,

$$\left[\frac{\partial \phi_1}{\partial r}\right]_{r=a} = -\sum_{n=0}^{\infty} \frac{(n+1)A_n}{a^{n+2}} P_n(\cos\theta).$$

Thus, we obtain

$$-m\sum_{n=0}^{\infty} \frac{na^{n-1}}{f^{n+1}} P_n(\cos\theta) = -\sum_{n=0}^{\infty} \frac{(n+1)A_n}{a^{n+2}} P_n(\cos\theta).$$

Hence

$$A_n = \frac{mn}{n+1}\frac{a^{2n+1}}{f^{n+1}}.$$

Thus

$$\phi_1 = m\sum_{n=0}^{\infty} \frac{na^{2n+1}}{(n+1)f^{n+1}}\frac{P_n(\cos\theta)}{r^{n+1}} = m\sum_{n=0}^{\infty} \frac{a^{2n+1}}{f^{n+1}}\frac{P_n(\cos\theta)}{r^{n+1}} - m\sum_{n=0}^{\infty} \frac{a^{2n+1}}{(n+1)f^{n+1}}\frac{P_n(\cos\theta)}{r^{n+1}}$$

$$= \frac{m}{a}\sum_{n=0}^{\infty}\left(\frac{a^2}{f}\right)^{n+1}\frac{P_n(\cos\theta)}{r^{n+1}} - \frac{m}{a}\sum_{n=0}^{\infty} \frac{a^{2n+2}}{(n+1)f^{n+1}}\frac{P_n(\cos\theta)}{r^{n+1}}$$

$$= \frac{mb}{a}\sum_{n=0}^{\infty} \frac{b^n}{r^{n+1}} P_n(\cos\theta) - \frac{m}{a}\sum_{n=0}^{\infty} \frac{a^{2n+2}}{(n+1)f^{n+1}}\frac{P_n(\cos\theta)}{r^{n+1}} = \frac{ma}{fr_2} - \frac{m}{a}\sum_{n=0}^{\infty} \frac{b^{n+1}}{(n+1)r^{n+1}} P_n(\cos\theta)$$

$$= \frac{ma/f}{r_2} - \frac{m}{a}\int_0^b d\chi \sum_{n=0}^{\infty} \frac{\chi^n}{r^{n+1}} P_n(\cos\theta)$$

Set,

$$\lambda = \frac{1}{r'} = \frac{1}{(r^2+\chi^2-2r\chi\cos\theta)^{1/2}} = \sum_{n=0}^{\infty} \frac{\chi^n}{r^{n+1}} P_n(\cos\theta).$$

Hence

$$\int_0^b \lambda d\chi = \int_0^b d\chi \sum_{n=0}^{\infty} \frac{\chi^n}{r^{n+1}} P_n(\cos\theta).$$

Therefore,

$$\phi_2 = \frac{ma/f}{r_2} - \frac{m}{a}\int_0^b \frac{d\chi}{r'}.$$

This shows that the required image consists of the source of strength $\frac{ma}{f}$ at the inverse point Q' and a line distribution of sink of strength $-\frac{m}{a}$ per unit length extending from the center to the inverse point.

2.5.3 Image of a doublet in front of sphere :

Let a doublet of strength μ be placed at A on the z-axis, where OA = f and OA′ = f + δf so that $m\delta f \to \mu$, where m is the strength of source and sink. Let B and B′ be the inverse points of A and A′ respectively with respect to the sphere. The image of m at A is $\frac{ma}{f}$ at B and a line distribution of sink of strength $-\frac{m}{a}$ per unit length from O to B. The image of – m at A′ is $-\frac{ma}{f+\delta f}$ at B′, that is $-\frac{ma}{f}+\frac{ma\delta f}{f^2}$ and a line source of strength $\frac{m}{a}$ per unit length from 0 to B′.

Compounding this image system, we get a doublet of strength $\frac{ma}{f}BB'$, a source $ma\frac{\delta f}{f^2}$ and a sink $-\frac{m}{a}BB'$, all ultimately at the inverse point. Since $OB = \frac{a^2}{f}$, so $BB' = \frac{a^2\delta f}{f^2}$ so that the source and sink cancel each other and there remains a doublet of strength $\frac{ma}{f}\cdot\frac{a^2\delta f}{f^2} = \frac{ma^3\delta f}{f^3}$ i.e., $\frac{\mu a^3}{f^3}$ at the inverse point in the opposite direction to the given doublet.

2.6 Illustrative Solved Examples

Example 1 :

Show that when a sphere of radius a moves with uniform velocity U through a perfect incompressible infinite fluid, the acceleration of a particle of the fluid at (r, 0) is

$$3U^2\left(\frac{a^3}{r^4}-\frac{a^6}{r^7}\right).$$

Solution :

Superimpose a velocity – U both to the sphere and the liquid. This reduces the sphere to rest and the velocity potential of the flow is given by (Article 'Liquid steaming past a fixed sphere')

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \tag{1}$$

Also

$$\dot{r} = -\frac{\partial\phi}{\partial r} = -U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \tag{2}$$

and

$$r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1 + \frac{a^3}{r^3}\right)\sin\theta. \tag{3}$$

Again, from (2), we have

$$\ddot{r} = U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\dot{\theta} - U\frac{3a^3}{r^4}\dot{r}\cos\theta$$

$$= U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\dot{\theta} + \frac{3a^4}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right)\cos^2\theta, \text{ by (2)}.$$

Clearly for a point (r, 0), the velocity is only along the direction of r and hence the acceleration will also be only along r.

Thus the required acceleration

$$= \ddot{r} \text{ only at } (r, 0)$$

$$= \frac{3a^2}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right), \text{ from (3) with } \theta = \dot{\theta} = 0$$

$$= 3U^2\left(\frac{a^3}{r^4} - \frac{a^6}{r^7}\right).$$

Example 2 :

A stream of water of greater depth is flowing with a uniform velocity U over a plane level bottom. A hemisphere of weight W in water and radius a, rests with its base on the bottom. Prove that the average pressure between the base of the hemisphere and the bottom is less than the fluid pressure at any point of the bottom at a great distance from the hemisphere if

$$U^2 = \frac{32W}{11\pi a^2 \rho}.$$

Solution :

Let water be flowing past a fixed hemisphere with velocity U along z-axis and (r, θ, ω) be the spherical polar co-ordinates of a point referred to the center of the hemisphere as the origin.

The velocity potential is given by

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \tag{1}$$

Then

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[U\left(1 - \frac{a^3}{r^3}\right)\cos\theta\right]_{r=a} = 0.$$

$$\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[-U\left(1 + \frac{a^3}{r^3}\right)\sin\theta\right]_{r=a} = -\frac{3}{2}U\sin\theta.$$

Let q be the velocity at any point of the boundary of the sphere r = a. Then, we have

$$q^2 = \left\{\left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\right\}_{r=a} = \frac{9}{4}U^2\sin^2\theta. \tag{2}$$

In steady motion in absence of external forces, the pressure at any point by Bernoulli's equation is given by

$$\frac{p}{\rho} + \frac{1}{2}q^2 = C. \tag{3}$$

But p = Π, q = U at infinity. So (3) gives

$$\frac{\Pi}{\rho} + \frac{1}{2}U^2 = C. \tag{4}$$

Subtracting (4) from (3), we obtain

$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2. \tag{5}$$

Using (2), the pressure p′ at any point P on the surface of the sphere r = a is given by

$$p' = \Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta. \tag{6}$$

Relation between (x, y, z) and (r, θ, ω) are given by

$$x = r\sin\theta\cos\omega,\ \ y = r\sin\theta\sin\omega;\ \ z = r\cos\theta. \tag{7}$$

Direction cosine of OP are (x/r, y/r, z/r) where OP = r = a. Using (2), direction cosine of OP are (sin θ cos θ, sin θ sin θ, cos θ).

Hence the component of p′ along x-axis is p′ sin θ cos ω.

Taking a sin θdω, adθ as an element on the surface of the hemisphere, the total thrust on the hemisphere due to water along OX

$$= \int_{\theta=0}^{\pi}\int_{\omega=-\pi/2}^{\pi/2} (p'\sin\theta\cos\omega)(a\sin\theta d\omega . ad\theta)$$

$$= a^2\int_0^{\pi}\int_{-\pi/2}^{\pi/2}\left[\Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2\sin^2\theta\right]\sin^2\theta\cos\omega d\theta d\omega \ \text{[using (1)]}$$

$$= 2a^2\int_0^{\pi}\left[\Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2\sin^2\theta\right]\sin^2\theta d\theta$$

$$= 2a^2\int_0^{\pi}\left[\left(\Pi + \frac{1}{2}\rho U^2\right) - \frac{9}{8}\rho U^2\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}\right]$$

$$= \pi a^2\left(\Pi - \frac{11\rho U^2}{32}\right).$$

Since there is a weight W on the base, the total thrust on the base

$$= \pi a^2\left(\Pi - \frac{11\rho U^2}{32}\right) + W.$$

Therefore,

$$\text{average pressure on the base} = \frac{\text{pressure on base}}{\text{area of the base}} = \Pi - \frac{11\rho U^2}{32} + \frac{W}{\pi a^2}.$$

Hence,

the average pressure < pressure at great distance

if

$$\Pi - \frac{11\rho U^2}{32} + \frac{W}{\pi a^2} < \Pi$$

i.e., if

$$U^2 > \frac{32W}{11\rho\pi a^2}.$$

Example 3 :

Incompressible fluid of density ρ, is contained between two rigid concentric spherical surfaces, the outer one of mass M_1 and radius a, the inner one of mass M_2 and radius b. A normal blow P is given to the outer surface. Prove that the initial velocities of the two containing surface (U for the outer and V for the inner) are given by the equations

$$\left\{M_1 + \frac{2\pi\rho a^3(2a^3 + b^3)}{3(a^3 - b^3)}\right\}U - \frac{2\pi\rho a^3 b^3}{a^3 - b^3}V = P$$

$$\left\{M_2 + \frac{2\pi\rho b^3(2b^3 + a^3)}{3(a^3 - b^3)}\right\}V = \frac{2\pi\rho a^3 b^3}{a^3 - b^3}U.$$

Solution :

As in article 'Moving Concentric Sphere', we have

$$\phi = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)r + \frac{(V-U)a^3b^3}{2r^2}\right]\cos\theta \qquad (1)$$

The normal blow P in the outer imparts velocity U to the outer and V to the inner spherical surface. Let ϖ_1, ϖ_2 be the impulsive pressure on an element ds of the boundary surface r = a and r = b respectively. Then

$$M_1 U = P - \iint \varpi_1 \cos\theta\, ds \text{ on } r = a \qquad (2)$$

and

$$M_2 U = P - \iint \varpi_2 \cos\theta \, ds \text{ on } r = b \tag{3}$$

On r = a, from (1)

$$\varpi_1 = (\rho\phi)_{r=a} = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)a + \frac{(V-U)ab^3}{2}\right]\cos\theta.$$

Hence (2) reduce to

$$M_1 U = P - \int_0^\pi \varpi_1 \cos\theta . a d\theta . 2\pi a \sin\theta$$

$$= P - \frac{2\pi a^3}{a^3 - b^3}\left[(Vb^3 - Ua^3)a + \frac{(V-U)ab^3}{2}\right] \times \int_0^\pi \cos^2\theta \sin\theta d\theta$$

$$= P - \frac{\pi a^3}{a^3 - b^3}\left[3Vb^3 - U(2a^3 + b^3)\right] \times \left(-\frac{2}{3}\right).$$

Therefore,

$$\left\{M_1 + \frac{2\pi\rho a^3(2a^3 + b^3)}{3(a^3 - b^3)}\right\}U - \frac{2\pi\rho a^3 b^3}{a^3 - b^3}V = P. \tag{4}$$

Again, on r = b

$$\varpi_2 = (\rho\phi)_{r=b} = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)b + \frac{(V-U)a^3 b}{2}\right]\cos\theta.$$

Hence (3) reduces to

$$M_2 V = -\int_0^\pi \varpi_2 \cos\theta . a d\theta . 2\pi b \sin\theta$$

$$-\frac{2\pi b^2}{a^3 - b^3}\left[(Vb^3 - Ua^3)b + \frac{(V-U)a^3 b}{2}\right] \times \int_0^\pi \cos^2\theta \sin\theta d\theta$$

$$= -\frac{2\pi b^3 \rho}{a^3 - b^3}\left[Vb^3 - Ua^3 + \frac{1}{2}a^3(U - V)\right] \cdot \left(-\frac{2}{3}\right).$$

Therefore,

$$\left\{M_2 + \frac{2\pi\rho b^3(2b^3 + a^3)}{3(a^3 - b^3)}\right\}V = \frac{2\pi\rho a^3 b^3}{a^3 - b^3}U.$$

Example 4 :

Prove that if two rigid surface of revolution one of which surrounds the other, are moving along their common axis with velocities U_1, U_2 and space between them filled with homogenous liquid, the momentum of the liquid is $M_2U_2 - M_1U_1$, where M_1, M_2 are the masses of liquid which either surface would contain.

Solution :

Let x-axis be taken as the axis of revolution. Due to symmetry, the moment of momentum of the liquid along the y-axis and z-axis is zero. The momentum of the liquid along x-axis is

$$\iiint \rho u dxdydz. \tag{1}$$

If ϕ be the potential at any point P(x, y, z) of the liquid, then $u = -\frac{\partial\phi}{\partial x}$ and so (1) becomes

$$-\iiint \rho \frac{\partial\phi}{\partial x} dxdydz. \tag{2}$$

the integration extends over the whole volume of the liquid.

Using the Green's theorem (2) can be re-written as

$$\iint x \frac{\partial\phi}{\partial x} ds. \tag{3}$$

where δn is an element of the outward normal at the element of the bounding surface δs.

Hence the momentum of the liquid along x-axis is

$$= \rho\iint_{inner} x\frac{\partial\phi}{\partial x} ds_1 + \rho\iint_{outer} x\frac{\partial\phi}{\partial x} ds_2$$

$$= -\rho\iint_{inner} xl_1U_1 ds_1 + \rho\iint_{outer} xl_2U_2 ds_2 \tag{4}$$

where l_1 and l_2 are cosines of the angles which the outer drawn normals at ds_1, ds_2 make with x-axis.

But $l_1 ds_1 = dxdy$ and $l_2 ds_2 = dydz$. So (4) reduces to :

The momentum of the liquid along x-axis

$$= -\rho\iint xU_1 dxdy + \rho\iint xU_2 dydz$$

$$= -U_1\iint \rho x dxdy + U_2\iint \rho x dydz$$

$$= M_2U_2 - M_1U_1,$$

where M_1, M_2 are the masses of the liquids which either surface would contain.

2.7 Model Questions

Short Questions :

1. Find the solution of Laplace's equation in spherical polar coordinates having axial symmetry.
2. Define Stokes' stream function.
3. Define source, sink and doublet. Hence find the velocity potential for each of them.

Broad Questions :

1. Introducing Stokes' stream function, discuss the irrotational axi-symmetric motion of an ideal liquid.
2. A solid moves along the axis of revolution OX with velocity U in a non-viscous liquid the motion of the liquid being symmetrical about OX and irrotational. Discuss the motion.
3. Find the expression for the velocity potential and the equation of stream lines for the irrotational motion of a non-viscous liquid at rest at infinity in which a sphere is moving with uniform velocity, the motion being symmetrical about z-axis.
4. Deduce the equation of motion of a sphere moving in an incompressible ideal fluid at rest at infinity with velocity U along the axis of z. Hence show that the effect of the presence of the liquid is to reduce the external force in the ratio $(\sigma - \rho) : (\sigma + \frac{1}{2}\rho)$, σ and ρ being the densities of the sphere and the liquid respectively.

5. Discuss the irrotation motion of an ideal liquid past a fixed sphere in a uniform stream. Hence find the equation of the lines of flow.

6. The region between two concentric spheres is filled with a homogeneous incompressible fluid, the surfaces of the spheres being subjected to given impulses in the z-direction so that the two spheres start to move with given velocities in the positive direction at the z-axis. Determine the resulting motion.

7. Find the image of a source (or sink or doublet) with respect to a rigid plane.

8. Find the image of a source (or sink or doublet) in front of a sphere.

Problems :

1. An infinite ocean of an incompressible perfect liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is disturbed by sphere, and the pressure in the liquid at a great distance from the obstacles is Π. Show that the thrust on that half of the sphere on which the liquid impinges is

$$\pi a^2 \cdot \left\{ \Pi - \frac{\rho U^2}{16} \right\}.$$

2. Find the pressure at any point of a liquid, of infinite extent and at rest a great distance, through which a sphere is moving under no external forces with constant velocity U, and show that the mean pressure over the sphere is in defect of the pressure Π at a great distance by $\frac{1}{4}\rho U^2$, it being supposed that Π is sufficiently large for the pressure everywhere to be positive, that is, that

$$\Pi > \frac{5}{8}\rho U^2.$$

3. Liquid of density ρ fills the space between a solid sphere of radius a and density ρ′ and a fixed concentric spherical envelope of radius b; prove that the work done by an impulse which starts the solid sphere with velocity V is

$$\frac{1}{3}\pi a^3 V^3 \left(2\rho' + \frac{2a^3 + b^3}{b^3 - a^3}\rho \right).$$

4. The space between two concentric spherical shells of radii a and b(a > b) is filled with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V in the same direction : prove that the resultant impulsive pressure on the inner shell is

$$\frac{2\pi\rho b^2}{3(a^3 - b^3)}\{3a^3U - (a^3 + 2b^3)V\}.$$

5. A sphere of radius a is made to move in incompressible perfect fluid with non-uniform velocity u along x-axis. If the pressure at infinity is zero, prove that at a point x in advance of the center

$$p = \frac{1}{2}\rho a^3\left\{\frac{\dot{u}}{x^2} + u^2\left(\frac{2}{x^3} - \frac{a^3}{x^6}\right)\right\}.$$

2.8 Summary

In this chapter, we have considered the three-dimensional irrotational motion of an ideal liquid with special reference to a sphere and a solid of revolution. Notion of source, sink, doublet and their images with respect to a rigid plane and a sphere has also been introduced.

Unit 3 □ Vortex Motion

Structure

3.0 Introduction

It is well known that for irrotational motion the velocity vector $\mathbf{q} = (u,v,w)$ can be represented in the form of the gradient of a velocity potential ϕ as

$$\mathbf{q} = \text{grad } \phi$$

or, in other words,

$$u = \frac{\partial \phi}{\partial x},\ v = \frac{\partial \phi}{\partial y},\ w = \frac{\partial \phi}{\partial z}. \tag{1}$$

The *vorticity* is defined to be a vector Ω = curl $\mathbf{q}$, whose components are

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2}$$

The above components vanish when the conditions (1) are satisfied. Thus, for an irrotational motion when $\mathbf{q} = \text{grad}\phi$,

$$\Omega = \text{curl grad } \phi = 0. \tag{3}$$

Conversely, if $\Omega = 0$, then with the aid of vector analysis, it can be shown that equation (1) will always hold. Thus, in irrotational motion, a velocity potential certainly exists.

This chapter will consist of investigation of such motions of a fluid for which the *vorticity vector* Ω is different from zero at least in some part of the fluid under consideration. We will call such motions as vortex motions of the fluid.

3.1 Vortex lines and Vortex tubes

A vortex line is a curve in the fluid such that its tangent at any point gives the direction of the local vorticity. Therefore, the equations of a vortex line have the form

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \tag{4}$$

where Ω_x, Ω_y, Ω_z are the components of the vorticity vector Ω. Note that, the above equations are analogous to the equations for a streamlines. Portions of the fluid bounded by vortex lines through every point of an infinitely small closed curves are called vortex filaments, or simply vortices. Vortex lines passing through any closed curve form a tubular surface, which is called a *vortex tube*. The fluid contained within such a tube constitutes what is called a vortex-filament. Let δS_1, δS_2 be two sections of a vortex tube and let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the unit normals to these sections drawn outwards from the fluid between them. Also, let δS be the curved surface of the vortex tube. Then, $\Delta S = \delta S_1 + \delta S_2 + \delta S$ = total surface area of the element. Let ΔV be the total volume contained in ΔS. Then

$$\int_{\Delta S} \mathbf{n}.\Omega dS = \int_{\Delta V} \mathrm{div}\Omega dV = 0,$$

since div $\Omega = 0$. Thus

$$\int_{\delta S_1} \mathbf{n}.\Omega dS = \int_{\delta S} \mathbf{n}.\Omega dS + \int_{\delta S_2} \mathbf{n}.\Omega dS = 0.$$

At each point of δS, $\mathbf{n}.\Omega = 0$, since Ω is tangential to the curved surface. Thus

$$(\mathbf{n_1}.\Omega)\delta S_1 + (\mathbf{n_2}.\Omega)\delta S_2 = 0$$

approximately to the first order (using the mean value theorem of integral calculus). This shows that $|\mathbf{n}.\Omega| dS$ is constant for every section δS of the vortex tube. Its value is called the *strength* of the vortex tube. A vortex tube whose strength is unity is called a *unit vortex tube*.

Some properties of vertices :

(1) Vortex lines and tubes move with the fluid.

Let C be any closed curve drawn on the surface of the vortex tube containing an area S of the tube and not embracing the tube. As the vortieity vectors are everywhere lying on the surface S, it follows that, $\mathbf{n}.\Omega = 0$. So the circulation Γ around C is given by

$$\int_\Gamma \mathbf{q}.d\mathrm{s} = \int_S \mathbf{n}.\Omega dS = 0.$$

After an interval of time, the same fluid particles form a new surface, say S'. According to Kelvin's theorem, the circulation around S' must also be zero. As this is true for any S, the component of vortieity normal to every element of S' must vanish, showing that S' must lie on the surface of the vortex tube. Hence, vortex lines and vortex tubes move with fluid.

(2) Vortex lines and tubes move with the fluid.

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(3) A vortex tube cannot originate or end within the fluid. It must either end at a solid boundary or form a closed loop (a 'vortex ring').

Suppose S is any closed surface containing a volume V. Then

$$\int_S \mathbf{n}.\Omega dS = \int_V \mathrm{div}\,\Omega dV = 0. \quad (5)$$

Equation (5) shows that the total strength of vortex tubes emerging from S is equal to that entering S. This means that *vortex lines and tubes cannot originate* or *terminate at internal points in a fluid.* They can only form closed curves or terminate on boundaries.

(4) Strength of a vortex tube remains constant for all time.

If C is a closed curve embracing once the vortex tube and if S denotes an area contained in C, then the circulation Γ of the fluid velocity **q** around the vortex tube is defined as

$$\Gamma = \oint_C \mathbf{q}.ds \tag{6}$$

Then, by Stokes' theorem

$$\Gamma = \int_S \mathbf{n}.\mathbf{q}dS. \tag{7}$$

Equation (7) shows that Γ is nothing but the strength of vortex tube with surface area S. Since for an inviscid fluid the circulation around any closed curve in the fluid moving along with the fluid, remains constant in time, therefore strength of the vortex also remains constant in time.

The above theorems are known as **Helmholtz's vortex theorems :**

We shall assume that the fluid is a single-valued function of time only.

3.2 Rectilinear Vortex

Consider a single tube whose cross-section is a circle of radius a and with its axis parallel to the axis of z surrounded by unbounded fluid. The motion is similar in all planes parallel to xy and it has no velocity along the axis of z. By making the area contained within the tube sufficiently small we see that the distribution producing such a flow must be uniform along the z-axis. Such a distribution along the z-axis is called a uniform *rectilinear* or *line vortex*. Thus if **q** = (u, v, w) be the velocity, then w = 0 and u, v are independent of z. If $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ be the vorticity vector, then

$$\Omega_x = 0, \Omega_y = 0, \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8}$$

The velocity components u, v are related to the stream function ψ by

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}. \tag{9}$$

Use of (9) in (8) gives

$$\Omega_z = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \tag{10}$$

Thus, ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \begin{cases} \Omega_x, & \text{on the vortex,} \\ 0, & \text{out side the vortex.} \end{cases} \tag{11}$$

Let P(r, θ) be any point outside the vortex. Since the motion outside the vortex is irrotational, the velocity potential ϕ exists and

$$\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{12}$$

holds, r, θ being polar coordinates. Since, in the region out side the vortex ψ is harmonic so we get

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{13}$$

If the motion is symmetric about the origin, ψ must be independent of θ. Then equation (13) reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0$$

giving

$$\psi = c \log r, \; c = \text{constant}. \tag{14}$$

Using the relation of ϕ and ψ given by (12) we get

$$\phi = -c\theta. \tag{15}$$

Thus the complex potential function w is given by

$$w = \phi + i\psi = -c\theta + ic \log r = ic \log z. \tag{16}$$

Let k be the circulation in the circuit enclosing the vortex. Then

$$k = \int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) r d\theta = 2\pi c$$

so that

$$c = \frac{k}{2\pi}$$

and hence w is given by

$$w = \frac{ik}{2\pi} \log z. \tag{17}$$

This is the complex potential due to a vortex of strength k placed at the origin. If the vortex be placed at $z_0 = x_0 + iy_0$, instead of (0, 0), then the complex potential w has the form

$$w = \frac{ik}{2\pi} \log(z - z_0). \tag{18}$$

Let $P(z) \equiv P(x, y)$ be another point in the fluid other than (x_0, y_0). Then distance r_0 between (x, y) and (x_0, y_0) is given by

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2. \tag{19}$$

From (18), we see that the stream function ψ is given by

$$\psi = \frac{k}{2\pi} \log r_0.$$

Thus,

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial r_0}\frac{\partial r_0}{\partial y} = -\frac{k}{2\pi} \cdot \frac{y - y_0}{r_0^2}$$

and

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r_0}\frac{\partial r_0}{\partial x} = \frac{k}{2\pi} \cdot \frac{x - x_0}{r_0^2}.$$

Thus the magnitude of the velocity **q** is given by

$$q = \left(u^2 + v^2\right)^{\frac{1}{2}} = \frac{k}{2\pi r_0}.$$

This is the velocity at any point P(x, y) due to presence of a vortex of strength k at (x_0, y_0).

Note :

If there be any number of vertices of strength k_s at z_s, s = 1, 2, 3, ..., then the complex potential at any point z in the fluid is given by

$$w = \frac{i}{2\pi}\sum_s k_s \log(z - z_s),$$

and the velocity components are given by

$$u = -\frac{1}{2\pi}\sum_s k_s \frac{(y - y_s)}{r_s^2} \text{ and } v = \frac{1}{2\pi}\sum_s k_s \frac{(x - x_s)}{r_s^2}$$

where

$$z_s = x_s + iy_s \text{ and } r_s^2 = (x - x_s)^2 + (y - y_s)^2.$$

Let (u_s, v_s) denote the velocity components of the vortex of strength k_s. Then

$$u_s = -\frac{1}{2\pi}\sum_{r \neq s} k_r \frac{(y_r - y_s)}{R_{r_s}^2} \text{ and } v_s = \frac{1}{2\pi}\sum_{r \neq s} k_r \frac{(x_r - x_s)}{R_{r_s}^2}$$

where

$$R_{r_s}^2 = (x_r - x_s)^2 + (y_r - y_s)^2.$$

Note that the expressions $\sum k_s u_s$ and $\sum k_s v_s$ will consist of pairs of terms of the forms

$$k_r \cdot \frac{k_s}{2\pi}\frac{(x_r - x_s)}{R_{r_s}^2} \text{ and } k_s \cdot \frac{k_r}{2\pi}\frac{(x_s - x_r)}{R_{r_s}^2}$$

and as such

$$\sum k_s u_s = 0 \text{ and } \sum k_s v_s = 0.$$

Hence, regarding k as a mass, the center of gravity of the vortex system, viz.

$$\bar{x} = \frac{\sum k_s u_s}{\sum k_s}, \quad \bar{y} = \frac{\sum k_s v_s}{\sum k_s}$$

remains stationary throughout the motion. Note that if $\sum k_s = 0$, the center $(\bar{x}, \bar{y})$ is at infinity.

3.3 Circular Vortex

Let there be a single cylindrical vortex tube, whose cross-section is a circle of radius a, surrounded by unbounded fluid.

The section of the vortex by the plane of the motion is a circle and the arrangement may therefore be referred to as a *circular vortex*.

3.3.1 Vortex pair

Consider the case of two vortices of strengths k_1 and k_2 at a distance r_0 apart. Let A, B be their centers, O, the center of the system. The point O divides AB in the ratio $k_2 : k_1$. The motion of each vortex as a whole is entirely due to the other, and is therefore always perpendicular to AB. Hence the two vortices remain ahvays at the same distance from one another and rotate with constant angular velocity about O which is fixed. The velocities at the two vortices at A and B are respectively $\dfrac{k_1}{2\pi r_0}$ and $\dfrac{k_2}{2\pi r_0}$. To obtain the angular velocity w of the system, we divide the velocity of the vortex A by the distance AO, where

$$AO = \frac{k_2}{k_1 + k_2} . AB = \frac{k_2 r_0}{k_1 + k_2} .$$

Therefore, the angular velocity is given by

$$\omega = \frac{\text{velocity of the vortex at A}}{AO} = \frac{k_1 + k_2}{2\pi r_0^2} .$$

If k_1, k_2 be of the same sign, i.e. if the direction of rotation in the two vortices be the same then O lies between A and B; otherwise O lies in AB or BA, produced. If $k_1 = -k_2$, O is at infinity. However, A, B move with equal velocities $\dfrac{k_1}{2\pi r_0}$ at right angles to AB, which remains fixed in direction. Such a combination of two equal and opposite vortices may be called a *vortex pair*.

3.3.2 Vortex doublet

Consider a vorlex pair, k at $ae^{i\alpha}$ and $-k$ at $-ae^{i\alpha}$ in the complex z-plane where $z = x + iy$. If we let $a \to 0$ and $k \to \infty$ so that $2ak = \mu$ is a finite constant, we get a vortex doublet of strength μ inclined at an angle α to the x-axis.

The direction of the doublet is determined from the vortex of negative rotation to that of positive rotation. The complex potential is

$$w = \lim_{a\to 0} \frac{ik}{2\pi}\{\log(z - ae^{i\alpha}) - \log(z + ae^{i\alpha})\}$$

$$= \lim_{a\to 0} \frac{ik}{2\pi}\left(-\frac{ae^{i\alpha}}{z} + \frac{a^2e^{2i\alpha}}{2z^2} - \cdots - \frac{ae^{i\alpha}}{z} - \frac{a^2e^{2i\alpha}}{2z^2} - \cdots\right) = -\frac{i\mu}{2\pi z}e^{2i\alpha}.$$

The stream function is $\psi = -\frac{\mu}{2\pi r}\cos(\alpha - \theta)$.

If, in particular, we take the vortex doublet to be at the origin and along the axis of y, we have $\psi = -\frac{\mu\sin\theta}{2\pi r}$. If we put $\frac{\mu}{2\pi} = Ub^2$, we obtain $\psi = -\frac{Ub^2\sin\theta}{r}$ which is the stream function for a circular cylinder of radius b moving with velocity U along the x-axis.

Thus the motion due to a circular cylinder is the snme as that due to a suitable vortex doublet placed at the center, and with its axis perpendicular to the direction of motion.

3.4 Infinite row of parallel rectilinear vortices

3.4.1 Single infinite row

Consider an infinite row of vortices each of strength k at the points 0, ±a, ±2a, ..., ±na, ... (as shown in figure 3.1).

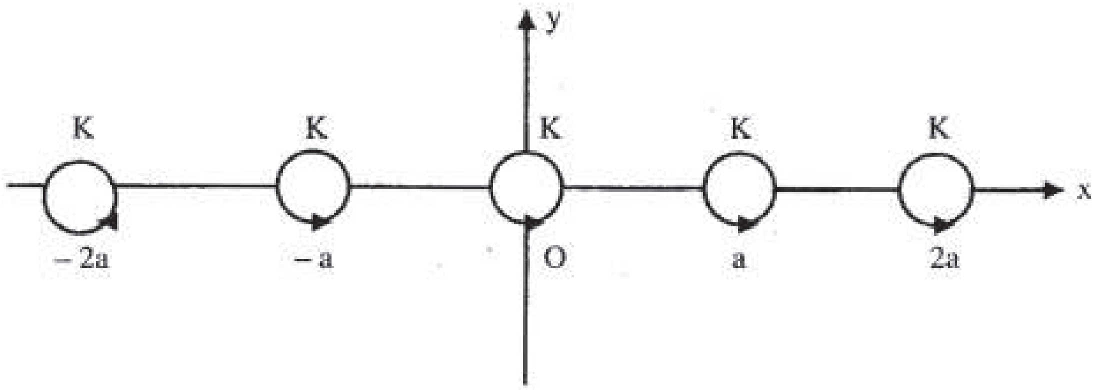

Figure 3.1

The complex potential of the (2n + 1) vortices nearest to the origin is

$$w_n = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\log(z-a) + \cdots + \frac{ik}{2\pi}\log(z-na)$$

$$+\frac{ik}{2\pi}\log(z+a) + \cdots + \frac{ik}{2\pi}\log(z+na)$$

$$= \frac{ik}{2\pi}\log\{z(z^2-a^2)(z^2-2^2a^2)\cdots(z^2-n^2a^2)\}$$

$$= \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\} + \frac{ik}{2\pi}\log\left\{\frac{a}{\pi}.a^2.2^2a^2\ldots n^2a^2\right\}.$$

The constant term may be omitted, so that we write

$$w_n = \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\}. \tag{20}$$

Now, sin x can be expressed as an infinite product in the form

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\cdots \tag{21}$$

Thus letting $n \to \infty$ in (20), we get by virtue of (21),

$$w = \frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right). \tag{22}$$

Consider the vortex at $z = 0$. Since its moton is due to the other vortices, the complex velocity of the vortex at the origin is given by

$$-\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z\right\}_{z=0} = -\frac{ik}{2\pi}\left(\frac{\pi}{a}\cdot\cot\frac{\pi z}{a} - \frac{1}{z}\right)_{z=0} = 0.$$

Thus the vortex at the origin is at rest. Similarly it can be shown that the remaining vortices are also at rest. Thus the vortex row induces no velocity in itself.

To determine the stream function we note that

$$w(z) = \phi + i\psi,\ \overline{w}(\overline{z}) = \phi - i\psi$$

so that from (22)

$$2i\psi = w(z) - \overline{w}(\overline{z}) = \frac{ik}{2\pi}\log\left\{\sin\frac{\pi z}{a}\sin\frac{\pi\overline{z}}{a}\right\},$$

$$\psi = \frac{k}{4\pi}\log\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right).$$

For large values of $\frac{y}{a}$, we neglect the term $\cos\frac{2\pi x}{a}$, for its modulus never exceeds unity, and therefore along the streamlines ψ = constant. Thus at a great distance from the row the stream lines are parallel to the row.

Again, if v_1, v_2 are the complex velocities at the points z, $\bar{z}$ respectively, we have

$$v_1 + v_2 = -\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=z} - \frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=\bar{z}}$$

$$= -\frac{ik}{2a}\cot\frac{\pi z}{a} - \frac{ik}{2a}\cot\frac{\pi \bar{z}}{a} = -\frac{ik}{2a}\frac{2\sin\frac{2\pi x}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}},$$

which is purely imaginary and tends to zero when y tends to infinity. Thus the velocities along the distant streamlines are parallel to the row but in opposite directions.

3.4.2 Infinite row of parallel rectilinear vortices (Karman Vortex Street)

This consists of two parallel infinite rows of the same spacing, say a, but of opposite vortex strengths k and – k, so arranged that each vortex of the upper row is directly above

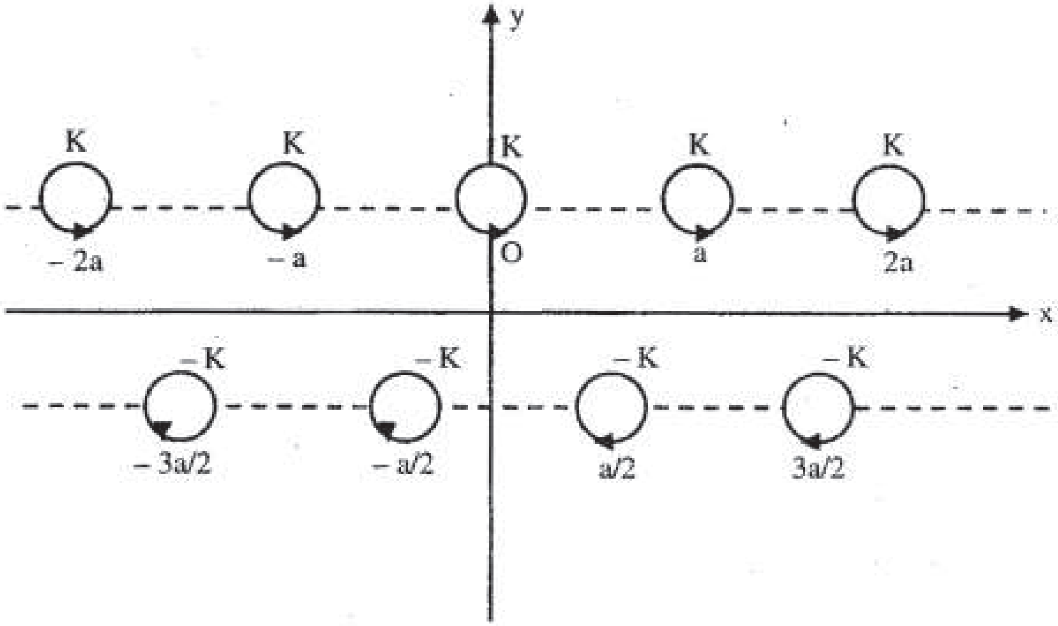

Figure 3.2

the mid point of the line joining two vortices of the lower row and vice-versa. Taking the configuration at time t = 0, we take the axes as shown in the figure 3.2, the x-axis being midway between and parallel to the rows which are at the distance b apart. At this instant the vortices in the upper row are at the points $ma + \frac{1}{2}ib$, and those in the lower row at the points $\left(m+\frac{1}{2}\right)a - \frac{1}{2}ib$, where $m = 0, \pm 1, \pm 2, \ldots$

The complex potential at the instant t = 0, by the preceding section is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{ib}{2}\right) + \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{a}{2} + \frac{ib}{2}\right).$$

Since neither row induces any velocity in itself, the velocity of vortex at $z = \frac{a}{2} - \frac{ib}{2}$ will be given by

$$-u_1 + iv_1 = \left[\frac{d}{dz}\frac{ik}{2\pi}\sin\frac{\pi}{a}\left(z - \frac{ib}{2}\right)\right]_{z=\frac{a}{2}-\frac{ib}{2}}$$

$$= \frac{ik}{2a}\cot\left(\frac{\pi}{2} - \frac{i\pi b}{a}\right) = -\frac{k}{2a}\tanh\frac{\pi b}{a}.$$

Thus the lower row advances with velocity

$$V = \frac{k}{2a}\tanh\frac{\pi b}{a},$$

and similarly the upper row advances with the same velocity. The rows will advance the distance a in time $\tau = \frac{a}{V}$ and the configuration will be the same after this interval as at the initial instant.

Note :

In a Karman vortex street, under the influence of some operation, all or certain of the vortices may experience small displacements. Then it is possible that with the passage of time the vortices will remain close to the positions which they would have had if they had not been subject to displacements. We then say that the motion is stable. If, however, the displaced vortices tend to move away from the position corresponding to unperturbed state, the motion will be called unstable. A necessary condition of stability for the Karman's vortex street is

$$\cosh \frac{b\pi}{a} = \sqrt{2}$$

so that b = 0.281a.

3.5 Illustrative Solved Examples

Example 1

If

$$u = \frac{ax - by}{x^2 + y^2}, \; v = \frac{ay + bx}{x^2 + y^2}, \; w = 0,$$

investigate the nature of motion of the liquid.

Solution :

Given

$$u = \frac{ax - by}{x^2 + y^2}, \; v = \frac{ay + bx}{x^2 + y^2}, \; w = 0. \tag{1}$$

From (1),

$$\frac{\partial u}{\partial x} = \frac{a(x^2 + y^2) - 2x(ax - by)}{(x^2 + y^2)^2} = \frac{ay^2 - ax^2 + 2bxy}{(x^2 + y^2)^2}$$

and

$$\frac{\partial v}{\partial y} = \frac{a(x^2 + y^2) - 2y(ay + bx)}{(x^2 + y^2)^2} = \frac{ax^2 - ay^2 - 2bxy}{(x^2 + y^2)^2}.$$

We see that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and hence the equation of continuity is satisfied by (1). Therefore (1) represents a two-dimensional motion and hence vorticity components are given by

$$\Omega_x = 0, \; \Omega_y = 0, \; \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2}$$

From (1),

$$\frac{\partial u}{\partial y}=\frac{-b(x^2+y^2)-2y(ax-by)}{(x^2+y^2)^2}=\frac{by^2-bx^2-2axy}{(x^2+y^2)^2}$$

and

$$\frac{\partial v}{\partial x}=\frac{b(x^2+y^2)-2x(ay+bx)}{(x^2+y^2)^2}=\frac{by^2-bx^2-2axy}{(x^2+y^2)^2}$$

so that $\Omega_z = 0$. Thus

$$\Omega_x = 0,\ \Omega_y = 0,\ \Omega_z = 0$$

showing that the motion is irrotational.

Example 2

Find the necessary and sufficient conditions that vortex lines may be at right angles to the streamlines.

Solution :

Streamlines and vortex lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \tag{1}$$

and

$$\frac{dx}{\Omega_x}=\frac{dy}{\Omega_y}=\frac{dz}{\Omega_z} \tag{2}$$

respectively. These will be at right angles, if

$$u\Omega_x = v\Omega_y = w\Omega_z. \tag{3}$$

But

$$\Omega_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\ \Omega_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\ \Omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{4}$$

Using (4), (3) may be written as

$$u\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+v\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+w\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0, \tag{5}$$

which is the necessary and sufficient condition that udx + vdy + wdz may be a perfect differential. So we may write

$$udx + vdy + wdz = \mu d\phi = \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right).$$

Thus the necessary and sufficient conditions that vortex lines may be at right angles to the streamlines are

$$u = \mu\frac{\partial\phi}{\partial x},\ v = \mu\frac{\partial\phi}{\partial y}, w = \mu\frac{\partial\phi}{\partial z}.$$

Example 3

When an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance 2b, prove that the streamlines relative to this system are given by the equation

$$\log\frac{x^2 + (y-b)^2}{x^2 + (y+b)^2} + \frac{y}{b} = C,$$

the origin being the midpoint of the line joining the two vortices, taken as the y-axis.

Solution :

Let there be two rectilinear vortices of strengths k and – k at $P_1(0, b)$ and $P_2(0, -b)$ respectively. Thus $P_1P_2 = 2b$, origin being the midpoint of P_1P_2 and y-axis being taken along P_1P_2. Thus we have a vortex pair which will move with a uniform velocity $k/2\pi P_1P_2$ or $k/4\pi b$ perpendicular to the line P_1P_2 (ie. along the x-axis). To determine the streamlines relative to the vortices, we must impose a velocity on the given system equal and opposite to the velocity $k/4\pi b$ of motion of the vortex pair. Accordingly, we add a term $\frac{kz}{4\pi b}$ to the complex potential of the vortex pair. Note that

$$-\frac{d}{dz}\left(\frac{kz}{4\pi b}\right),$$

and hence the term added is justified. So, for the case under consideration, the complex potential is given by

$$w = \phi + i\psi = \frac{ik}{2\pi}\log(z - ib) - \frac{ik}{2\pi}\log(z + ib) + \frac{kz}{4\pi b}.$$

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi}\log\left[x^2 + (y-b)^2\right] - \frac{k}{4\pi}\log\left[x^2 + (y+b)^2\right] + \frac{kz}{4\pi b}$$

or

$$\psi = \frac{k}{4\pi}\left[\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}\right].$$

Hence the required relative streamlines are given by ψ = constant, i.e.,

$$\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}=C.$$

Example 4

If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $8\pi^2a^2/(n-1)k$, and find the velocity of any part, of the liquid.

Solution :

Let us take the origin as the center of the circle of radius a and the x-axis along the line $\theta = 0$. Suppose that n rectilinear vortices each of strength k be situated at points $z_m = a\exp^{2\pi i m/n}$, m = 0, 1, 2, ..., n – 1 on the circumference of the circle. Then the complex potential due to these n vortices is given by

$$w = \frac{ik}{2\pi}\sum_{m=0}^{n-1}\log(z-a\exp^{2\pi i m/n})$$

$$= \frac{ik}{2\pi}\prod_{m=0}^{n-1}(z-a\exp^{2\pi i m/n}) = \frac{ik}{2\pi}\log(z^n-a^n).$$

Now the fluid velocity q at any point out of all the n vortices is given by

$$q = \left|-\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{z^{n-1}}{z^n-a^n}\right| = \left|\frac{kn}{2\pi}\frac{z^n-1}{z^n-a^n}\right|.$$

Again the velocity induced at the point z = a, by the other vortices is given by the complex potential

$$w' = \frac{ik}{2\pi}\log(z^n-a^n) - \frac{ik}{2\pi}\log(z-a)$$

so that

$$w - \frac{dw}{dz} = \frac{ik}{2\pi} \log\left(z^{n-1} + z^{n-1}a + \cdots + za^{n-1} + a^{n-1} \right).$$

Hence

$$\left(\frac{dw'}{dz} \right)_{z=a} = \frac{ik}{2\pi} \frac{(n-1)+(n-2)+\cdots+2+1}{na} = \frac{ik(n-1)}{4\pi a}$$

or

$$u_1 - iv_1 = \left(\frac{dw'}{dz} \right)_{z=a} = -\frac{ik(n-1)}{4\pi a}$$

so that $u_1 = 0$ and $v_1 = \frac{k(n-1)}{4\pi a}$. If q_r and q_θ be the radial and transverse velocity components of the velocity at $z = a$, then we have $q_r = 0$ and $q_\theta = \frac{k(n-1)}{4\pi a}$. Due to symmetry of the problem, it follows that each vortex moves with the same transverse velocity $\frac{k(n-1)}{4\pi a}$. Hence the required time T is given by

$$T = \frac{2a\pi}{\frac{k(n-1)}{4\pi a}} = \frac{8\pi^2 a^2}{(n-1)k}.$$

3.6 Model Questions

Short Questions :

1. Define : Vortex (or vortex filament), vortex lines, vortex tubes, rectilinear vortex, circular vortex, vortex pair, vortex doublet.
2. Prove the following results :
   (a) Vortex lines and tubes move with the fluid.
   (b) Strength of a vortex tube is constant alongth the length and for all time.
   (c) Vortex lines and tubes cannot originate or terminate at internal points in a fluid.
3. Find the expression for the angular velocity of a pair of vortices.

4. Show that the motion due to a circular cylinder is the same as that due to a suitable vortex doublet placed at the centre, with its axis perpendicular to the direction of motion.

Broad Questions :

1. Find the complex potential due to n vortices of strengths $k_1, k_2, ..., k_n$. Hence find the velocity components of the vortex of strength k_s ($1 \leq s \leq n$). Also, show that the centre of gravity of the vortex system remains at rest.
2. Discuss the motion of an infinite row of vortices, each of strength k situated in a straight line at equal distance apart. Hence show that, at a great distance from the row, the stream lines are parallel to the row.
3. What is meant by Karman Vortex street? Discuss the motion of rectilinear vortices lying on such a street. Also deduce the condition of stability of Karman Vortex street.

Problems :

1. In example1 find the velocity potential of the system.
2. If $udx + vdy + wdz = d\theta + \lambda d\chi$, where θ, λ, χ are function of x, y, z, t, prove that the the vortex lines at any time are the lines of intersection of the surfaces

$$\lambda = \text{constant and } \chi = \text{constant.}$$

3. If in the solved example-3, the vortices are of the same strength and the spin is in same sense both, show that the relative streamlines are given by

$$\log(r^4 + b^4 - 2b^2r^2\cos 2\theta) - (r^2/2b)^2 = \text{constant},$$

θ being measured from the join of the vortices, the origin being its middle point. Show also that the surfaces of equipressure at any instant are given by $r^4 + b^4 - 2b^2r^2\cos 2\theta = \lambda(r^2 \cos 2\theta + a^2)$.

4. Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move round the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$.

5. If (r_1, θ_1), (r_2, θ_2), ..., be polar coordinates at time t of a system of rectilinear vortices of strength $k_1, k_2, ...$, prove that

$$\sum kr^2 = \text{constant and } \sum kr^2\dot{\theta} = (1/2\pi)\sum k_1 k_2 .$$

6. An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. Show also that, if α be the radius of a vortex, the amount of flow between two vortex and the next is $(k/\pi) \log \cot (\pi\alpha/2a)$.

7. An infinite street of linear parallel vortices is given as : $x = ra$, $y = b$, strength k; $x = ra$, $y = -b$, strength $= -k$, where r is any positive or negative integer or zero. Prove that if the liquid at infinity is at rest, the street moves as a whole in the direction of its length with the speed $(k/2a) \coth (2\pi b/a)$.

Unit 4 □ Surface Waves

Structure

4.0 Introduction

It is a matter of common observation that if a pebble is thrown into a pond, then some disturbance travels radially over the water surface. Such a disturbance is known as water waves. Also, if a piano is played in a room, then sound wave is spread there. The energy extracted from the sun is transmitted through waves in ether. All these are examples of wave motion. Thus we nitice two distinguished features : (a) *energy is propagated at distant points* and (b) *the disturbance travels through the medium without any transference of the medium itself.* In fact, these two properties do exist whatever be the medium which transmits the waves.

4.1 General Expression for Wave Motion

Consider an arbitrary disturbance ϕ moving along the positive x-axis with velocity c. Thus ϕ is a function of x and t, say $\phi = f(x, t)$. The curve when $t = 0$, i.e., $\phi = f(x)$ is known as *wave profile.* If the disturbance moves without changing its shape, then the wave profile has moved through a distance ct in the positive direction of x-axis at time t. If the distance measured from the new origin $x = ct$ be denoted by ξ so that $x - ct = \xi$, then the equation of the wave profile referred to the new origin is $\phi = f(\xi)$, in other words, referred to the original origin, it is

$$\phi = f(x - ct). \tag{1}$$

Similarly, the equation $\phi = f(x + ct)$ represents the same disturbances moving in the negative direction of x-axis with velocity c.

4.2 Wave Motion in Liquid

A wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above some chosen fixed horizontal plane varies.

Taking the axis of x to be horizontal and the axis of y to be vertically upwards, a motion in which the equation of the vertical section of the free surface at time t is of the form

$$y = a \sin (mx - nt), \tag{2}$$

where a, m, n are constants, is called a Simple harmonic progressive wave. Since (2) can be written in the form

$$y = a \sin m\left(x - \frac{nt}{m}\right), \tag{3}$$

this shows that the wave profile $y = a \sin mx$ at $t = 0$ moves with velocity n/m (= c, say) in the positive x-direction. c is called the *velocity of propagation* of the wave. When a = 0 the profile of the liquid is y = 0, which is the *mean level*. The quantity a is called the *amplitude* of the wave and measures the maximum departure of the actual free surface from the mean level. The points $C_1, C_2, \ldots$ of maximum elevation are known as *crests* and the points $T_1, T_2, \ldots$ of maximum depression are known as *troughs*. The distance between successive crests is called the *wave-length* and is denoted by λ. Thus

$$\lambda = \frac{2\pi}{m}.$$

Again the nature of the free surface (2) remains unchanged by replacing t by t + 2π/n. The time T = 2π/n is known as the *period* of the wave. The reciprocal of the period is known as the *frequency* it denotes the number of oscillations per second. The angle mx – nt is known as *phase angle*. If the equation of wave motion be y = a sin(mx – nt + ∈), then ∈ is called the *phase* of the wave.

4.3 Standing or Stationary Waves

Two simple harmonic progressive waves of the same amplitude, wave length and period travel in opposite directions are given by the surface elevation

$$\eta_1 = \frac{1}{2} a \sin(mx - nt), \; \eta_2 = \frac{1}{2} a \sin(mx + nt).$$

By the principle of superposition, the resulting surface elevation is represented by the equation

$$\eta = \eta_1 + \eta_2 = 2a \sin mx \cos nt.$$

A motion of this type is called a *stationary or standing wave*. An any instant the equation represents a sine curve but the amplitude 2a cos nt varies continuously.

The points of intersection of the curve with the x-axis are fixed points called *nodes*. When a progressive train of waves represented by η_1 impinges on a fixed vertical barrier and is there reflected (η_2), the resulting disturbance when a steady state is reached consists of stationary waves.

Such waves can, for example, be generated by tilting slightly a rectangular vessel containing water and then restoring it to the level position. The water level at each end of the vessel then moves up and down the vertical faces which are loops. Conversely a progressive wave can be regarded as due to the superposition of two standing waves.

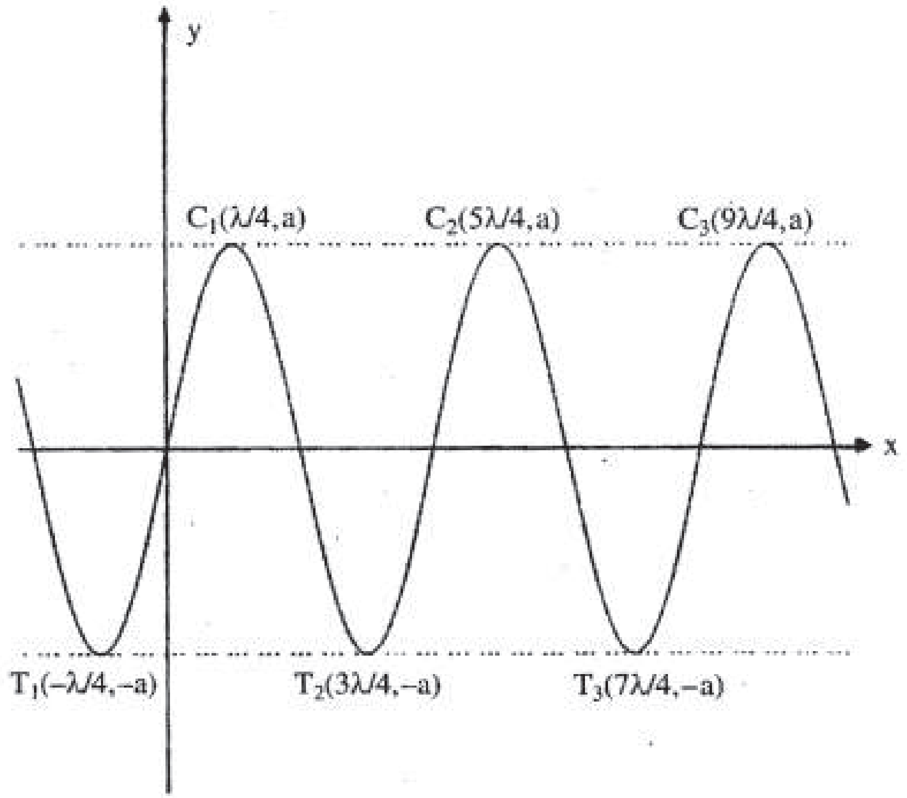

Figure 4.1

4.4 Surface Waves

Such waves occur at and near the free surface of an unbounded sheet of liquid where the depth is considerable compared to the wave length. For these waves the vertical acceleration is comparable with the horizontal acceleration, and so we consider forces both in horizontal and vertical directions.

The x-axis is taken in the undisturbed surface in the direction of propagation of the waves and the y-axis vertically upwards. Taking the motion to be irrotational, incompressible and two-dimensional, the velocity potential ϕ exits such that

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \qquad (4)$$

throughout the liquid, and

$$\frac{\partial\phi}{\partial n}=0 \tag{5}$$

at a fixed boundary.

The pressure can be obtained from the Bernoulli's equation

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-gy-\frac{1}{2}q^2+C(t). \tag{6}$$

The free surface is a surface of equipressure p = constant, hence on the free surface

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0, \tag{7}$$

where u and v are the velocity components on the free surface in x and y directions respectively. But

$$u=-\frac{\partial\phi}{\partial x},\ v=-\frac{\partial\phi}{\partial y}, \tag{8}$$

and at the free surface the relation (7) becomes

$$\frac{\partial p}{\partial t}-\frac{\partial\phi}{\partial x}\frac{\partial p}{\partial x}-\frac{\partial\phi}{\partial y}\frac{\partial p}{\partial y}=0. \tag{9}$$

Let the motion be so small that the squares of small quantities may be omitted. Again, without loss of generality we may include C(t) in ϕ and then substitute the value of p from (6) in (9) to get

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial t}-\frac{\partial\phi}{\partial y}\left(\frac{\partial^2\phi}{\partial y\partial t}-g\right)=0. \tag{10}$$

Neglecting the second and third terms which are of the same order as q^2, we obtain

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial y}=0. \tag{11}$$

This condition holds at the free surface.

If η is the elevation of the free surface at time t above the point whose abscissa is x, the equation of the free surface is given by

$$y-\eta(x,t)=0. \tag{12}$$

But we know that if

$$F(x, y, t) = y - \eta(x, t) = 0$$

is a boundary surface, then we must have

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

or

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - v = 0. \tag{13}$$

But $\frac{\partial \eta}{\partial t}$ is $\dot{\eta}$, and $\frac{\partial \eta}{\partial x}$ is the tangent of the slope of the free surface which by hypothesis is small so that the second term can be neglected and the equation becomes

$$\dot{\eta} = v = -\frac{\partial \phi}{\partial y} \tag{14}$$

at the free surface.

Hence in a wave motion in which the squares of the velocities can be neglected, the velocity potential must be a solution of Laplace's equation which makes

$$\frac{\partial \phi}{\partial n} = 0$$

at a fixed boundary and satisfies (11) and (14) at the free surface of the liquid.

4.4.1 Progressive waves on the surface of water

Consider the propagation of simple harmonic waves of the type

$$\eta = a\sin(mx - nt) \tag{15}$$

at the surface of water of uniform depth h, either of unlimited extent or contained in a channel with parallel vertical sides at right angle to the ridges and hollows.

If we assume that there is a solution of the form

$$\phi = f(y)\cos(mx - nt)$$

and substitute in (4) we obtain

$$\frac{\partial^2 f}{\partial y^2} - m^2 f = 0, \tag{16}$$

so that

$$f(y) = Ae^{my} + Be^{-my},$$

and

$$\phi = (Ae^{my} + Be^{-my}) \cos(mx - nt).$$

This value of f must be satisfy (15), i.e.

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{when } y = -h.$$

Hence

$$Ae^{-mh} = Be^{mh} = \frac{1}{2}C, \text{ say,}$$

so that

$$\phi = C \cosh m(y + h) \cos(mx - nt). \tag{17}$$

Again if we substitute this value in the surface condition (8) putting y = 0, we get

$$n^2 = gm \tanh mh. \tag{18}$$

Now let c = n/m and λ = 2π/m denote velocity of propagation and the wave length respectively. Then we get

$$c^2 = \frac{g}{m} \tanh mh = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}. \tag{19}$$

We now determine the constant C of (17) in terms of the amplitude a of the wave. Using (15) and (17), the boundary condition (14) gives

$$-na = -mC \sinh mh,$$

so that

$$\phi = \frac{na}{m} \frac{\cosh m(y + h)}{\sinh mh} \cos(mx - nt), \tag{20}$$

or, using (18) we obtain

$$\phi = \frac{ga}{n} \frac{\cosh m(y + h)}{\cosh mh} \cos(mx - nt). \tag{21}$$

The path of the particle

If (x, y) be the coordinates of a particle relative to its mean position, neglecting the squares of small quantities we may write

$$\frac{dx}{dt} = -\frac{\partial\phi}{\partial x} = na\frac{\cosh m(y+h)}{\sinh mh}\sin(mx - nt),$$

$$\frac{dy}{dt} = -\frac{\partial\phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh}\cos(mx - nt).$$

Integrating above two equations, we get

$$x = a\frac{\cosh m(y+h)}{\sinh mh}\cos(mx - nt),$$

$$y = a\frac{\sinh m(y+h)}{\sinh mh}\sin(mx - nt);$$

so that the particle describes the ellipse

$$\frac{x^2}{\cosh^2 m(y+h)} + \frac{y^2}{\sinh^2 m(y+h)} = \frac{a^2}{\sinh^2 mh}$$

about its mean position. For a given particle mx – nt plays the part of the eccentric angle in the ellipse; so that the eccentric angle increases at a uniform rate, as in an orbit described under a central force varying as the distance.

4.4.2 Progressive waves on a deep water

If the depth h of the water be sufficiently great in comparison with λ for e^{-mh} to be neglected, then the constant B = 0 in the above case, so that we have instead of (17)

$$\phi = Ae^{my}\cos(mx - nt) \tag{22}$$

and instead of (18)

$$n^2 = gm \tag{23}$$

or,

$$c^2 = \frac{g\lambda}{2\pi}. \tag{24}$$

Also if

$$\eta = a \sin(mx - nt)$$

is the free surface we get from (14)

$$na = mA,$$

so that

$$\phi = \frac{na}{m} e^{my} \cos(mx - nt),$$

$$\phi = \frac{ga}{n} e^{my} \cos(mx - nt). \tag{25}$$

Following the case **4.4.1** we get in this case for the displacement of a particle from its mean position

$$\begin{aligned} x &= ae^{my} \cos(mx - nt), \\ y &= ae^{my} \sin(mx - nt), \end{aligned} \tag{26}$$

and the path of the particle is a circle

$$x^2 + y^2 = a^2 e^{2my},$$

described with uniform angular velocity n, which in this case is equal to $(gm)^{\frac{1}{2}}$ or $\left(\frac{2\pi g}{\lambda}\right)^{\frac{1}{2}}$.

4.4.3 Stationary waves on the surface of water

Consider a stationary wave of the type

$$\eta = a \sin mx \cos nt. \tag{27}$$

The velocity potential for a system of stationary waves can be deduced from **4.4.1** by regarding the system as the result of the superposition of two such trains of waves as we have just been considered moving in opposite directions as explained in **Section-4.3.**

Then we shall have

$$\phi = \frac{na}{m} \frac{\cosh m(y + h)}{\sinh mh} \sin mx \sin nt, \tag{28}$$

or,

$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\sin mx \sin nt \tag{29}$$

for ϕ satisfies (4) and (5), and η and ϕ together satisfy (14).

It is not necessary to regard standing waves as a case of superpositon of progressive waves. We might investigate this form for ϕ independently starting with the assumption

$$\phi = f(y)\sin mx \sin nt.$$

For standing waves in deep water, as in **4.4.2,** equations (28) and (29) take the forms

$$\left.\begin{aligned} \phi &= \frac{na}{m}e^{my}\sin mx \sin nt, \\ \phi &= \frac{ga}{n}e^{my}\sin mx \sin nt. \end{aligned}\right\} \tag{30}$$

Path of the particles :

In this case we have

$$\dot{x} = -\frac{\partial\phi}{\partial x} = -na\frac{\cosh m(y+h)}{\sinh mh}\cos mx \sin nt,$$

$$\dot{y} = -\frac{\partial\phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh}\cos mx \sin nt,$$

so that, by integration

$$x = a\frac{\cosh m(y+h)}{\sinh mh}\cos mx \cos nt,$$

and

$$y = a\frac{\sinh m(y+h)}{\sinh mh}\sin mx \cos nt.$$

Hence

$$\frac{y}{x} = \tanh m(y+h)\tan mx,$$

and since this is independent of t, the motion of each particle is rectilinear, the direction varying from vertical beneath the crests and troughs $\left(mx = (n+\frac{1}{2})\pi\right)$, to horizontal beneath the nodes ($mx = n\pi$).

4.5 The Energy of the Progressive Waves

Kinetic Energy **:** The kinetic energy possessed by the liquid (per unit thickness), stretching between two vertical plane situated at a distance of one wave length apart and perpendicular to the direction of flow, is known as the kinetic energy of the progressive wave.

Considering a train of progressive waves at the surface of water of depth h, given by

$$\eta = a\sin(mx - nt) \tag{31}$$

and

$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt). \tag{32}$$

Since the motion is irrotational, the kinetic energy is given by

$$T = -\frac{1}{2}\rho\int \phi\frac{\partial\phi}{\partial n}ds, \tag{33}$$

δn being normal drawn into the liquid and integration being performed along the profile of a wave length. In this case, we get kinetic energy

$$T = \frac{1}{2}\rho\int_0^\lambda \left[\phi\frac{\partial\phi}{\partial y}\right]_{y=0} dx$$

$$= \frac{1}{2}\rho g a^2 \int_0^\lambda \cos^2(mx - nt)\,dx$$

$$= \frac{1}{2}\rho g a^2 \lambda.$$

Potential Energy :

The potential energy due to the elevated liquid in a wave length (the energy being calculated relative to the undisturbed state) is known as the potential energy of a progressive wave.

Let us calculate the potential energy of liquid between two vertical planes parallel to the direction of propagation at unit distance apart. Then, for a single wave length, the potential energy is given by

$$V = \frac{1}{2}\rho \int_0^{\lambda} \eta^2 \, dx$$

$$= \frac{1}{4}\rho g a^2 \lambda ,$$

as $\lambda = 2\pi/m$.

Total energy per wave length is

$$= T + V$$

$$= \frac{1}{2}\rho g a^2 \lambda .$$

Hence it follows that the total energy per wave length is half kinetic energy and potential energy.

The energy of the stationary waves :

The energy of stationary waves may be calculated in the same way. Thus if we take

$$\eta = a \sin mx \cos nt$$

and

$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \sin mx \sin nt .$$

We find for the potential energy of a wave length

$$V = \frac{1}{4} g a^2 \rho \lambda \cos^2 nt ,$$

and for the kinetic energy

$$T = \frac{1}{4} g a^2 \rho \lambda \sin^2 nt .$$

Total energy per wave length at any time

$$= T + V$$

$$= \frac{1}{4} g a^2 \rho \lambda .$$

The amounts of kinetic and potential energy change continuously with the time.

4.6 Group Velocity

A local disturbance of the surface of still water will give rise to a wave which can be analyzed into a set of simple harmonic components each of different wave-length. We have seen that the velocity of propagation depends upon the wave-length and so the waves of different wave-lengths will be gradually sorted out into groups of waves of approximately the same wave-length. In the case of water waves, the velocity of the group is, in general, less than the velocity of the individual waves composing it. What happens in this case is that the waves in front pass out of the group and new waves enter the group from behind. The energy within the group remains the same.

We now study the properties of such a group. To this end we examine the disturbance due to the superposition of two simple harmonic waves of the same amplitude and slightly different wave lengths,

$$\eta_1 = a\sin(mx - nt),$$

$$\eta_2 = a\sin\{(m+\delta m)x - (n+\delta n)t\}.$$

The resulting disturbance will be

$$\eta = \eta_1 + \eta_2$$

$$= 2a\cos\frac{1}{2}(x\delta m - t\delta n)\sin(mx - nt)$$

$$= A\sin(mx - nt) \tag{34}$$

where

$$A = 2a\cos\frac{1}{2}(x\delta m - t\delta n). \tag{35}$$

Equation (34) shows that the resulting disturbance is a progressive sine wave whose amplitude A is not constant but is itself varying as a wave of velocity

$$c_g = \frac{\delta n}{\delta m}. \tag{36}$$

This velocity is known as the *group velocity*.

Since the velocity of propagation of a single wave is

$$c = \frac{n}{m},$$

we have

$$c_g = \frac{dn}{dm} = \frac{d}{dm}(cm) = c + m\frac{dc}{dm}. \tag{37}$$

But $\lambda = \frac{2\pi}{m}$ so that

$$\frac{d\lambda}{dm} = -\frac{2\pi}{m^2}. \tag{38}$$

Then we get

$$c_g = c + m\frac{dc}{d\lambda}\frac{d\lambda}{dm} = c - \lambda\frac{dc}{d\lambda}. \tag{39}$$

For the case of waves on the surface of liquid of depth h, we have

$$c^2 = \frac{g}{m}\tanh mh. \tag{40}$$

From (37) and (40), we have

$$c_g = c\left(1 + \frac{m}{2c^2}\frac{dc^2}{dm}\right)$$

$$= \frac{1}{2}c\left(1 + \frac{2mh}{\sinh 2mh}\right) \tag{41}$$

so that the ratio of the group velocity to the wave velocity is given by

$$\frac{c_g}{c} = \frac{1}{2} + \frac{mh}{\sinh 2mh}$$

or,

$$c_g = \frac{1}{2}c\left(1 + \frac{2mh}{\sinh 2mh}\right). \tag{42}$$

When h is small compared with the wave length, this ratio is unity, so that group velocity for shallow water is equal to the wave velocity. Also as h increases to infinity the ratio decreases to $\frac{1}{2}$; or the group velocity for deep sea waves is half the wave velocity.

4.7 Rate of Transmission of Energy in Simple Harmonic Surface Waves

In a simple harmonic train of surface waves, energy crosses a fixed vertical plane perpendicular to the direction of propagation at an average rate equal to group velocity.

Proof :

Consider vertical section of the liquid at right angle to the direction of propagation. Then the rate of transmission of energy is calculated by determining the rate at which the pressure on one side of the chosen section is doing work on the liquid on the other side. Now, the velocity potential is given by

$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \cos(mx - nt). \qquad (43)$$

Again neglecting squares of small quantities the variable part of the pressure is given by

$$\delta p = \rho \frac{\partial \phi}{\partial t}. \qquad (44)$$

and the horizontal velocitv is

$$u = -\frac{\partial \phi}{\partial x}. \qquad (45)$$

Hence the work done in unit time or the energy carried across unit width of the section is

$$W = -\int_{-h}^{0} \delta p \frac{\partial \phi}{\partial x} dy$$

$$= \frac{g^2 a^2 \rho m}{n} \frac{\sin^2(mx - nt)}{\cosh^2 mh} \int_{-h}^{0} \cosh^2 m(y+h) dy$$

$$= \frac{g^2 a^2 \rho m}{n} \frac{\sin^2(mx - nt)}{\cosh^2 mh} \left(\frac{\sinh 2mh}{4m} + \frac{h}{2} \right), \qquad (46)$$

and since

$$n^2 = gm \tanh mh,$$

this may be written as

$$W = \frac{1}{2} g\rho a^2 \frac{n}{m}\left(1 + \frac{2\,mh}{\sinh 2\,mh}\right)\sin^2 (mx - nt). \tag{47}$$

The mean value of the expression (47) over a complete period or any number of complete periods, or any interval that is so long compared to a period that the part corresponding to the fractional part of a period can be neglected in comparison with the whole, is

$$W = \frac{1}{4} g\rho a^2 \frac{n}{m}\left(1 + \frac{2\,mh}{\sinh 2\,mh}\right). \tag{48}$$

But the group velocity c_g is given by

$$c_g = \frac{1}{2} c\left(1 + \frac{2\,mh}{\sinh 2\,mh}\right). \tag{49}$$

Since $\frac{n}{m} = c$, then from (48) and (49) we get

$$W = \frac{1}{2}\left(\frac{1}{2} g\rho a^2\right) c_g. \tag{50}$$

Since $\frac{1}{2} g\rho a^2$ is the whole energy per unit length at any instant. Hence the energy is transmitted at a rate equal to the group velocity.

4.8 Progressive Waves Reduced to a Case of Steady Motion

In any case in which waves propagate in one direction only without change of shape, the problem of determining the velocity of propagation can be simplified as follows : Impose on the whole liquid a velocity equal and opposite to the velocity of propagation of the waves. Then the wave profile having the same relative velocity as before becomes fixed in space and the problem becomes one of steady motion. We now illustrate this technique by means of the following two cases :

Case-I : *Progressive waves on the surface of water*

Let progressive waves move on the surface on the channel of uniform depth h and having parallel vertical walls. Let the progressive waves move towards the positive direction of x-axis with velocity c without change of form. Impose on the whole liquid a velocity c in the negative direction of x-axis. The wave form having the same relative velocity as before becomes fixed in space and the problem becomes one of steady motion. As the problem is a two-dimensional one it only remains to determine suitable expressions for the velocity potential and stream function so that the free surface and the bottom of the liquid may satisfy the conditions for stream lines.

Consider the complex potential

$$w = cz + P \cos mz - iQ \sin mz,$$

or,

$$\phi + i\psi = c(x + iy) + P\cos m(x + iy) - iQ\sin m(x + iy).$$

It gives

$$\left.\begin{aligned} \phi &= cx + (P\cosh my + Q\sinh my)\cos mx, \\ \psi &= cy - (P\sinh my + Q\cosh my)\sin mx. \end{aligned}\right\} \tag{51}$$

Since ϕ and ψ given by (51) satisfy Laplace's equation, they represent a possible motion.

For the bottom to be a stream line we must have ψ is constant when $y = -h$ so that

$$-P \sinh mh + Q \cosh mh = 0.$$

Hence the expressions (51) may be written as

$$\left.\begin{aligned} \phi &= cx + A\cosh m(y + h)\cos mx, \\ \psi &= cy - A\sinh m(y + h)\sin mx. \end{aligned}\right\} \tag{52}$$

Let the free surface be a simple curve

$$\eta = a \sin mx.$$

Then from (4) the stream line $\psi = 0$ produces

$$ca - A \sinh mh = 0. \tag{53}$$

neglecting squares of small quantities.

Again, the formula for pressure is

$$\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} = \text{constant}. \tag{54}$$

At the free surface

$$y = a \sin mx$$

this becomes

$$\frac{p}{\rho} + ga \sin mx + \frac{1}{2} c^2 \{1 - 2\, ma \coth mh \sin mx\} = \text{constant}, \tag{55}$$

neglecting a^2.

But p is constant at the free surface. Hence (55) holds if the coefficient of sin mx vanishes, i.e.

$$g = c^2 m \coth mh,$$

or,

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}. \tag{56}$$

Case-II : *Progressive waves on a deep water*

For this case (when $h \to \infty$) we consider

$$\phi = cx + Ae^{my} \cos mx, \tag{57}$$

and

$$\psi = cy - Ae^{my} \sin mx \tag{58}$$

with a free surface

$$\eta = a \sin mx. \tag{59}$$

The free surface is the stream line $\psi = 0$, if

$$ca = A, \tag{60}$$

so that

and
$$\left.\begin{aligned} \phi &= cx + cae^{my} \cos mx, \\ \psi &= cy - cae^{my} \sin mx. \end{aligned}\right\} \tag{61}$$

The formula for the pressure

$$\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} = \text{constant}$$

becomes

$$\frac{p}{\rho}+gy+\frac{1}{2}c^2\{1-2mae^{my}\sin mx+m^2a^2e^{2my}\}=\text{constant}. \tag{62}$$

If we neglect the last term on the left, this equation may be written as

$$\frac{p}{\rho}+y(g-mc^2)+mc\psi=\text{constant}. \tag{63}$$

This equation not only gives

$$c^2=\frac{g}{m} \tag{64}$$

at the free surface, but also shows that, if $c^2=\frac{g}{m}$, the pressure is constant along each stream line. It follows that the solution contained in (56) and (64) can be applied to the case of any number of liquids of different densities arranged one above the other in horizontal strata including the case of liquid of continuously varying density since there is no limit to the thinness of a stream, the only limitations being that the upper surface is free and the total depth infinite.

4.9 Waves at the Common Surface of Two Liquids

Suppose a liquid of density ρ' and depth h' to be moving with velocity V' over another liquid of density ρ and depth h moving in the same direction with velocity V, the liquids being bounded above and below by two fixed horizontal planes.

Let c be the velocity of propagation of oscillatory waves at the interface of the two liquids in the direction in which the liquids are moving. Let the x-axis be in this direction in the undisturbed interface and y-axis vertically upwards. Let us make the motion steady by superposing on the whole mass the velocity $-c$ thereby bringing the wave form to rest in space.

The velocity and stream function for the lower liquid moving with the velocity $-(V-c)$ in the negative direction of x-axis and given by

$$\left.\begin{aligned}\phi&=-(V-c)x+A\cosh m(y+h)\cos mx,\\ \psi&=-(V-c)x-A\sinh m(y+h)\sin mx.\end{aligned}\right\} \tag{65}$$

Similarly expression for upper liquid may be deduced from (65) by replacing V by V′ and h by − h′. Thus we get

$$\left.\begin{aligned}\phi' &= -(V'-c)x + A'\cosh m(y-h')\cos mx,\\ \psi' &= -(V'-c)x - A'\sinh m(y-h')\sin mx.\end{aligned}\right\} \tag{66}$$

These expression for ψ and ψ' clearly make the boundaries $y = -h$, $y = h'$ stream lines; and if $\eta = a \sin mx$ gives the displacement of the common surface and the liquids do not separate this must be a stream line for both surfaces. We can satisfy this condition by taking the stream line to be $\psi = \psi' = 0$, which gives

$$\left.\begin{aligned}-(V-c)a - A\sinh mh &= 0,\\ -(V'-c)a + A'\sinh mh' &= 0,\end{aligned}\right\} \tag{67}$$

neglecting the squares of small quantities.

From Bernoulli's equations, we obtain

$$\left.\begin{aligned}\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} &= \text{constant},\\ \frac{p'}{\rho'} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi'}{\partial x}\right)^2 + \left(\frac{\partial\phi'}{\partial y}\right)^2\right\} &= \text{constant}.\end{aligned}\right\} \tag{68}$$

But at the interface

$$y = \eta = a \sin mx.$$

Then (68) gives (neglecting a^2)

$$\left.\begin{aligned}\frac{p}{\rho} + ga\sin mx + \frac{1}{2}(V-c)^2\{1 - 2\,ma\coth mh\sin mx\} &= \text{constant},\\ \frac{p'}{\rho'} + ga\sin mx + \frac{1}{2}(V'-c)^2\{1 - 2\,ma\coth mh'\sin mx\} &= \text{constant}.\end{aligned}\right\} \tag{69}$$

Since the pressure is continuous across the interface, putting $p = p'$ in above equations, subtracting and then equating to zero the coefficient of sin mx, we obtain

$$g(\rho-\rho') = (V-c)^2\, m\rho\coth mh + (V'-c)^2\, m\rho'\coth mh'. \tag{70}$$

This equation determines the velocity of propagation c of waves of length $\frac{2\pi}{m}$ at the common surface of two streams whose velocities are V, and V′; but it may also be regarded as the condition for stationary waves at the common surface of two streams whose velocities are $V - c$ and $V' - c$.

It should be noticed that in any such case as the above, even when V and V′ are both zero, the tangential velocities on opposite sides of the surface of separation are different so that this surface constitutes a vortex sheet.

4.9.1 Waves at the interface of two liquids with upper surface free

Another case of interest is that in which the surface of the upper liquid is free; *e.g.* a layer of oil upon water or of fresh water upon salt water.

Let a liquid of density ρ′ and depth h′ lie over another liquid of density ρ and depth h and let both the liquids to be at rest save for wave motion. We assume a common velocity of wave propagation c at the free surface of the upper liquid and at the common surface and reverse this velocity on the whole mass so that the motion becomes steady. We may take

$$\psi = cy - A \sinh m(y + h) \sin mx, \tag{71}$$

in the lower liquid, and

$$\psi' = cy - (B \cosh my + C \sinh my) \sin mx \tag{72}$$

in the upper liquid.

From (71), it easily follows that the bottom $y = -h$ is a stream surface $\psi = -ch$. Let the common surface be given by

$$\eta = a \sin mx, \tag{73}$$

it is also the stream surface $\psi = \psi' = 0$, if

$$ca - A \sinh mh = 0, \tag{74}$$

and

$$ca - B = 0. \tag{75}$$

Also the free surface

$$y = h' + b \sin mx \tag{76}$$

is a stream surface ψ' = constant if

$$cb - (B \cosh mh' + C \sinh mh') = 0. \tag{77}$$

From the Bernoulli's equation for the lower and upper liquids respectively, we have

$$\left.\begin{aligned}\frac{p}{\rho}+gy+\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}=\text{constant},\\ \frac{p'}{\rho'}+gy+\frac{1}{2}\left\{\left(\frac{\partial\psi'}{\partial x}\right)^2+\left(\frac{\partial\psi'}{\partial y}\right)^2\right\}=\text{constant}.\end{aligned}\right\} \quad (78)$$

Substituting from (71) and (72), using that A, B and C are of order a, neglecting squares of small quantities and equating the values of p and p′ at the common interface, we get

$$ga(\rho-\rho')-cm(\rho A\cosh mh-\rho' C)=0, \quad (79)$$

and using (74), (75) and (76), this gives

$$g(\rho-\rho')=c^2 m\left\{\rho\coth mh+\rho'\coth mh'-\rho'\frac{b}{a\sinh mh'}\right\}. \quad (80)$$

Then using the fact that p′ is constant at the free surface we get

$$gb = cm(B\sinh mh' + C\cosh mh'), \quad (81)$$

and from (74), (75) and (77) we obtain

$$g=c^2 m\left(\coth mh'-\frac{a}{b\sinh mh'}\right). \quad (82)$$

The elimination of the ratio a : b from (80) and (82) gives the equation for c, viz.

$$c^4 m^2(\rho\coth mh\coth mh'+\rho')-c^2 m\rho g(\coth mh+\coth mh')+g^2(\rho-\rho')=0 \quad (83)$$

and the ratio of the amplitudes of the waves is given from (82) by

$$\frac{b}{a}=\frac{c^2 m}{c^2 m\cosh mh'-g\sinh mh'}. \quad (84)$$

From (83) we see that there are two possible velocities of propagation for a given wave length, provided $\rho > \rho'$.

In the particular case in which the lower liquid is deep we put coth mh = 1. The roots of (83) are then

$$c^2=\frac{g}{m} \text{ and } c^2=\frac{g}{m}\frac{\rho-\rho'}{(\rho\coth mh'-\rho')}.$$

The ratio of the amplitudes of the upper and lower waves in the two cases are

$$e^{mh'} \quad \text{and} \quad -\left(\frac{\rho}{\rho'}-1\right)e^{-mh'}.$$

4.10 Long Waves of Small Elevation

These types of waves arise when the wave length of oscillations is much greater than the depth of the liquid and the disturbance affects the motion of the whole of the liquid.

For simplicity, consider the case of waves travelling in a straight canal of depth y_0 of uniform section. Take the x-axis is the direction of the length of the canal and y-axis vertically upwards and let η be the elevation of the free surface above the equilibrium level at the point whose abscissa is x at time t. We shall consider the case when the wave length λ is large so that (y_0/λ) is very small as well as (η/y_0) and $(d\eta/dx)$.

Then, so far as vertical forces are concerned we may regard the liquid to be in equilibrium and take the pressure at any point as the statical pressure due to the depth below the free surface. Thus the pressure p at a point (x, y) is given by

$$p - p_0 = g\rho(y_0 + \eta - y), \tag{85}$$

where p_0, supposed constant, is the pressure above the liquid. Hence we have

$$\frac{\partial p}{\partial x} = g\rho\frac{\partial \eta}{\partial x} \tag{86}$$

which is independent of y. Thus the horizontal acceleration of an element depends on the difference of pressure at its ends, i.e. $\frac{\partial p}{\partial x}dx$ so that the horizontal acceleration of all points in the same vertical cross-section remains the same. Consequently, those points which are once in a vertical plane always remain there.

We now consider a small horizontal cylinder PP′ of liquid of length dx′ and cross-section α, the difference of pressure at its ends being $g\rho\frac{\partial \eta}{\partial x'}dx'$. Also, if x be the abscissa of the vertical plane of particles through P in its equilibrium position and ξ be the horizontal displacement of this plane of particles, then $x' = x + \xi$ so that the horizontal acceleration is $\frac{\partial^2 \xi}{\partial t^2}$. The equation of motion is, therefore,

$$\rho \alpha dx' \frac{\partial^2 \xi}{\partial t^2} = -g\rho\alpha \frac{\partial \eta}{\partial x'} dx'$$

or,

$$\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \eta}{\partial x'} \tag{87}$$

Assuming the motion to be small and squares y small quantities can be neglected, we have from (87), by putting $x' = x + \xi$

$$\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \eta}{\partial x} \tag{88}$$

To form the equation of continuity, we suppose that A is the area of cross-section of the canal and b is the breadth of the surface. Then, in equilibrium position, the volume of liquid containing between the planes x and x + dx is Adx. Also, at time t, the distance between the bounding planes of the liquid is $dx + \frac{\partial \xi}{\partial x} dx$ and the area of the cross-section is $A + b\eta$. Thus

$$(A + b\eta)\left(dx + \frac{\partial \xi}{\partial x} dx\right) = Adx$$

or,

$$A \frac{\partial \xi}{\partial x} + b\eta = 0 \tag{89}$$

where we have neglected product of small quantities. Thus, from (88) we obtain

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{gA}{b} \frac{\partial^2 \xi}{\partial x^2}$$

whose solution is

$$\xi = f(x - ct) + F(x + ct) \tag{90}$$

where $c^2 = gA/b$. Equation (90) represents two waves travelling in opposite directions with velocity $c = (gA/b)^{\frac{1}{2}}$.

For a canal of rectangular cross-section of depth h, the wave velocity is $(gh)^{\frac{1}{2}}$ which is half the depth of the liquid.

The elevation η is given by (89) and (90) as

$$\eta = -\frac{A}{b} f'(x - ct) - \frac{A}{b} F'(x + ct) \tag{91}$$

Also, the particle velocity is

$$\dot{\xi} = -cf'(x - ct) + cF'(x + ct). \tag{92}$$

4.11 Capillary Waves

Let there be an interface between two liquids, like water in contact with air. This the interface will not be a constant pressure surface unless it is a plane surface. Since free surface is a curved surface, so waves would be effected due to a surface tension or energy per unit area due to capillary forces, the difference of the pressure on opposite sides of the surface is given by

$$T\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

where ρ and ρ' are the principal radii of curvature of the surface.

In the case of two-dimensional waves we have $\rho' = \infty$ and, if η denote the elevation,

$$\frac{1}{\rho} = -\frac{d^2\eta}{dx^2},$$

neglecting squares of small quantities. So if δp, δp' denote the variable parts of the pressure below and above the surface, we have

$$T\frac{d^2\eta}{dx^2} + \delta p - \delta p' = 0 \tag{93}$$

as the surface condition.

4.11.1 Capillary waves in a channel of uniform depth

Let us use the method of **Section-4.9,** reducing the problem to one of steady motion by superposing a velocity $-c$ on the whole mass, where c is the velocity of propagation. We have

$$\psi = cy - A \sinh m(y + h) \sin mx, \tag{94}$$

and for the free surface

$$\eta = a \sin mx, \tag{95}$$

provided

$$ca - A \sinh mh = 0. \tag{96}$$

Using these in the Bernoulli's equation, the variable part of the pressure is given by

$$\frac{\delta p}{\rho} + ga \sin mx + \frac{1}{2} c^2 \; (1 - 2\, am \coth mh \sin mx \,) = \text{constant}, \tag{97}$$

where the terms containing a^2 have been neglected. Now if we suppose that pressure on the upper side of the interface is constant, then $\delta p' = 0$ in (93) and so (93) reduces to

$$\delta p = -T \frac{d^2 \eta}{dx^2}$$

$$= T\, am^2 \sin mx. \tag{98}$$

Substituting this value in the last equation and equating to zero the coefficient of sin mx, we get

$$c^2 = \left(\frac{g}{m} + \frac{Tm}{\rho} \right) \tanh mh. \tag{99}$$

When h is large compared to the wave length this becomes

$$c^2 = \frac{g}{m} + \frac{Tm}{\rho}. \tag{100}$$

4.12 Illustrative Solved Examples

Example 1

When simple harmonic waves of length λ are propagated over the surface of deep water, prove that, at a point whose depth below the undisturbed surface is h, the pressure at the instants when the disturbed depth of the point is $h + \eta$ bears to the undisturbed pressure at the same point the ratio

$$1+\frac{\eta}{h}e^{-2\pi h/\lambda} : 1,$$

atmospheric pressure and surface tension being neglected.

Solution :

For deep water, the velocity potential is given by

$$\phi = \frac{na}{m}e^{my}\cos(mx - nt), \tag{1}$$

therefore

$$\frac{\partial\phi}{\partial t} = \frac{an^2}{m}e^{my}\sin(mx - nt). \tag{2}$$

Also

$$\eta = a\sin(mx - nt)\ c^2 = \frac{n^2}{m^2} = \frac{g}{m}.$$

So (2) becomes

$$\frac{\partial\phi}{\partial t} = g\eta e^{my}. \tag{3}$$

Pressure at any point within the water is given by

$$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + gy = C \text{ (a constant)}. \tag{4}$$

When $y = 0$, $p = 0$, $\frac{\partial\phi}{\partial t} = 0$ so

$$C = 0$$

and hence (4) gives

$$p = \rho \frac{\partial \phi}{\partial t} - g\rho y$$

or,

$$p = g\rho\eta e^{my} - g\rho y, \quad \text{by (3)}, \tag{5}$$

Therefore disturbed pressure p_1 when $y = -h$ is given by

$$p_1 = \rho g\eta e^{-mh} + \rho gh$$

$$= \rho gh\left(1 + \frac{\eta}{h} e^{-mh}\right) \tag{6}$$

and undisturbed pressure p_2 at a depth h is given by

$$p_2 = \rho gh. \tag{7}$$

Therefore

$$p_1 : p_2 = \left(1 + \frac{\eta}{h} e^{-mh}\right) : 1$$

$$= \left(1 + \frac{\eta}{h} e^{-2\pi h/\lambda}\right) : 1, \quad (\text{since } m = 2\pi/\lambda).$$

Example 2

Shew that, if the velocity of the wind is just great enough to prevent the propagation of waves of length λ against it, the velocity of propagation of waves with the wind is

$$2c\left\{\frac{\sigma}{(1+\sigma)}\right\}^{\frac{1}{2}},$$

where σ is the specific gravity of air and c is the wave velocity when no air is present.

Solution :

If V, V′ be the velocities of the lower and upper of two liquids of densities ρ, ρ' and depths h, h′, then

$$g(\rho - \rho') = m[(V - c_1)^2 \rho \coth mh + (V' - c_1)^2 \rho' \coth mh']. \tag{1}$$

Given $\frac{\rho'}{\rho} = \sigma$. Since the sea is at rest, V = 0 and h and h′ both $\to \infty$. Hence (1) reduces to

$$g(1-\sigma) = m\left[c_1^2 + (V' - c_1)^2 \sigma\right]. \qquad (2)$$

If no wind is present, V′ = 0, then

$$c_1 = c.$$

Therefore from (2),

$$g(1-\sigma) = m(c^2 + c^2\sigma)$$
$$= mc^2(1+\sigma). \qquad (3)$$

When there is no wave, $c_1 = 0$. From (2),

$$g(1-\sigma) = mV'^2\sigma. \qquad (4)$$

Now from (2),

$$g(1-\sigma) = m\left(c_1^2 + V'^2\sigma + c_1^2\sigma - 2V'c_1\sigma\right)$$

or,

$$mV'^2\sigma = m\left(c_1^2 + V'^2\sigma + c_1^2\sigma - 2V'c_1\sigma\right), \quad \text{using (4)}$$

or,

$$c_1^2(1+\sigma) - 2V'c_1\sigma = 0$$

$$V' = \frac{c_1(1+\sigma)}{2\sigma}. \qquad (5)$$

Putting this value of V′ in (4), we get

$$g(1-\sigma) = m\frac{c_1^2(1+\sigma)^2}{4\sigma^2}$$

or,

$$mc^2(1+\sigma) = m\sigma\frac{c_1^2(1+\sigma)^2}{4\sigma^2}$$

or,

$$c_1^2 = \frac{4\sigma}{1+\sigma} c^2,$$

so that

$$c_1 = 2c\left(\frac{\sigma}{1+\sigma}\right)^{\frac{1}{2}}.$$

Example 3

If there be two liquids in a straight channel of uniform section, of densities ρ_1, ρ_2 and depths l_1, l_2, shew that the velocity c of propagation of long waves is given by the equation

$$\left(\frac{c^2}{l_1 g} - 1\right)\left(\frac{c^2}{l_2 g} - 1\right) = \frac{\rho_1}{\rho_2},$$

where $\rho_2 > \rho_1$, and it is assume that the liquids do not mix.

Solution :

Proceeding as in **4.9.1** with $\rho = \rho_2$, $\rho' = \rho_1$, $h = l_2$, $h' = l_1$, we get from (25), of **4.9.1**,

$$c^4 m^2(\rho_2 \coth ml_2 \coth ml_1 + \rho_1) - c^2 gm\rho_2 (\coth ml_2 + \coth ml_1) + g^2(\rho_2 - \rho_1) = 0.$$

But for long waves, m is small and so we have

$$\coth ml_1 = \frac{1}{ml_1}$$

and

$$\coth ml_2 = \frac{1}{ml_2}$$

approximately. Therefore

$$\frac{c^4 \rho_2}{l_1 l_2} + \rho_1 m^2 c^4 - c^2 g\rho_2 \left(\frac{1}{l_1} + \frac{1}{l_2}\right) + g^2 (\rho_2 - \rho_1) = 0.$$

But for long waves,

$$m = \frac{2\pi}{\lambda}$$

is small. So neglecting m^2, we get

$$\frac{c^4}{l_1 l_2 g^2} - \frac{c^2}{g}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + 1 = \frac{\rho_1}{\rho_2}$$

or,

$$\left(\frac{c^2}{l_1 g} - 1\right)\left(\frac{c^2}{l_2 g} - 1\right) = \frac{\rho_1}{\rho_2}.$$

Example 4

Prove that

$$w = A\cos\frac{2\pi}{\lambda}(z + ih - ct)$$

is the complex potential for the propagation of simple harmonic surface waves of small high on water of depth h, the origin being in the undisturbed free surface. Express A in terms of the surface oscillations.

Solution :

We have for the progressive waves on the surface of water

$$\phi = \frac{ag}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt). \tag{1}$$

Since

$$\cos m(x + iy) = \cos mx \cosh my - i \sin mx \sinh my,$$

we take

$$\psi = \frac{ag}{n}\frac{\sinh m(y+h)}{\cosh mh}\sin(mx - nt). \tag{2}$$

Therefore

$$\begin{aligned} w &= \phi + i\psi \\ &= \frac{ag}{n}\frac{\cos((mx - nt) + im(y+h))}{\cosh mh} \\ &= \frac{ag}{n}\frac{\cos(m(x+iy) + imh - nt)}{\cosh mh} \end{aligned}$$

$$= \frac{ag}{n} \frac{\cos m\left(z + ih - \frac{nt}{m}\right)}{\cosh mh}$$

$$= \frac{ag}{n} \frac{\cos \frac{2\pi}{\lambda}(z + ih - ct)}{\cosh mh}.$$

Since

$$m = \frac{2\pi}{\lambda},\ c = \frac{n}{m}.$$

Therefore

$$w = A \cos \frac{2\pi}{\lambda}(z + ih - ct)$$

where

$$A = \frac{ag}{n \cosh mh} = \frac{ag}{mc \cosh mh}.$$

4.13 Model Questions

Short Questions :

1. Justify, by examples, the statement 'waves are means of propagation of energyy without any conspicuous movement of particles'.
2. What is meant by wave profile? Find the equation of the wave profile at any instant of time referred to a given origin.
3. Define : Simple harmonic progressive wave, standing (stationary) wave, surface wave, group velocity, capillary wave, long wave.
4. Show that a progressive wave can be regarded as due to the superposition of two standing waves.
5. Deduce the surface condition for capillary wave.

Broad Questions :

1. Deduce the condition at the free surface of an unbounded sheet of liquid for two-dimensional irrotational motion. Hence obtain the same if the motion be small.

Also show that in a wave motion in which the square of the velocities can be neglected, the velocity potential satisfies Laplace's equation and its normal derivative vanishes at a fixed boundary.

2. Discuss the motion of progressive waves (i) on the surface of water (ii) in deep water. Hence find the path of the particles in each case.

3. Deduce the expressions for the kinetic and potential energies of the progressive wave.

4. Discuss the motion of stationary waves (i) on the surface of water, (ii) in deep water. Hence find the path of the particles in each case.

5. Find the expression for the group velocity for waves on the surface of liquid of finite depth. Hence show that the group velocity for shallow water is equal to the wave velocity but that for deep sea waves is half the wave velocity.

6. Show that in a simple harmonic train of surface waves, energy crosses a fixed vertical plane perpendicular to the direction of propagation at an average rate equal to group velocity.

7. Find the rate of transmission of energy in simple harmonic surface waves.

8. Consider waves propagating in one direction without change of shape. Show how the problems of propagation of surface waves (i) on the surface of water and (ii) in a deep water, can be reduced to the problems of steady motion.

9. Discuss the motion of oscillatory waves at the interface of two liquids.

10. Discuss the motion of waves at the interface of two liquids with free upper surface.

11. What is meant by long wave? Show that for such waves, the points which lie once in a vertical plane always remain there.

12. Deduce the equations of motion and continuity for long waves. Hence find the solution of the equation of motion and interpret the result. Analyse the results for a canal of rectangular cross-section of given depth.

13. Discuss the motion of capillary waves in a channel of uniform depth.

Problems :

1. Let a shallow trough be filled with oil and water, and let the depth of the water be k and its density ρ_1, and the depth of the oil h and its density ρ_2. Then shew that if g be gravity, and v the velocity of propagation of long waves.

$$\frac{v^2}{g} = \frac{1}{2}(h+k) + \frac{1}{2}\left\{(h-k)^2 + \frac{4hk\rho_2}{\rho_1}\right\}^{\frac{1}{2}}.$$

Note that there may be slipping between the two fluids.

2. Two fluids of densities ρ_1, ρ_2 have a horizontal surface of separation but are otherwise unbounded. Shew that when waves of small amplitude are propagated at their common surface, the particles of the two fluids describe circles about their mean positions; and that at any point of the surface of separation where the elevation is η, the particles on either side have a relative velocity $\frac{4\pi c\eta}{\lambda}$.

3. If a channel of rectangular section contain a depth h of liquid of density ρ on which is superposed a depth h′ of liquid of density ρ', the free surface of the latter being exposed to constant atmospheric pressure, prove that the velocities of propagation of waves of length $2\pi/m$ are given by $c^2 = \frac{gu}{m}$, where

$$\rho(u \coth mh - 1)(u \coth mh' - 1) = \rho'(1 - u^2).$$

4. Two-dimensional waves of length $2\pi/m$ are produced at the surface of separation of two liquids which are of densities ρ, $\rho'(\rho > \rho')$ and depths h, h′ confined between two fixed horizontal planes. Prove that, if the potential energy is reckoned zero in the position of equilibrium, the total energy of the lower liquid is to that of the upper in the ratio

$$\rho((2\rho - \rho')\coth mh + \rho' \coth mh') : \rho'((\rho - 2\rho')\coth mh' - \rho \coth mh).$$

5. A channel, of infinite length and rectangular section, is of uniform depth h and breadth b in one part but changes gradually to uniform depth h′ and breath b′ in another part. An infinite train of simple harmonic waves travelling in one direction

only is propagated along the channel. Prove that, if a, a′ are the heights and $2\pi/m$, $2\pi/m'$ the lengths of the waves in the two uniform portions,

$$m \tanh mh = m' \tanh mh',$$

and

$$\frac{a^2 b}{\cosh^2 mh}(\sinh 2mh + 2mh) = \frac{a'^2 b'}{\cosh^2 m'h'}(\sinh 2m'h' + 2m'h').$$

4.14 Summary

The conception of surface waves relating to progressive and standing waves has been introduced. A sketch of long waves and capillary waves are also noted.

Unit 5 □ Viscous Flow

Structure

5.0 Introduction

So far we have considered the motion of an ideal or non-viscous fluid, that is the fluid which is incapable of exerting shearing (i.e. tangential) stress on any surface with which it is in contact. We now proceed to introduce the fluid motion for which the normal and the shearing stresses will be taken into account. The resulting equations, known as Navier-Stokes' equations are of fundamental importance and what else follows will be based on these equations.

5.1 Viscous Incompressible Flow : Navier-Stokes' Equations

It has already been seen (see study Material PG(MT)05 : Group-B, Page-122) that for incompressible viscous fluid, Navier-Stokes equation of motion is given by

$$\frac{\partial \vec{v}}{\partial t}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p+\nu\nabla^{2}\vec{v} \tag{1}$$

where $\vec{v}$ is the velocity vector; $\vec{F}$, the external force; ρ, the fluid density; p, the pressure and ν is the kinematic coefficient of viscosity.

Let us now consider some deductions from the equation (1).

Vorticity transport equation

We rewrite the equation (1) in the form

$$\frac{\partial \vec{v}}{\partial t}+\vec{\nabla}\left(\frac{1}{2}v^{2}\right)+\vec{w}\times\vec{v}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p+\nu\nabla^{2}\vec{v}$$

where $\vec{w}=\vec{\nabla}\times\vec{v}$ represents the vorticity vector. Assuming conservative nature of external forces so that $\vec{F}=-\vec{\nabla}\chi$, χ being potential function, we have from the above equation

$$\frac{\partial \vec{v}}{\partial t}+\vec{w}\times\vec{v}=-\vec{\nabla}\left(\chi+\frac{1}{2}v^{2}+\frac{p}{\rho}\right)+\nu\nabla^{2}\vec{v}.$$

Taking curl of both sides, it follows that

$$\frac{\partial \vec{w}}{\partial t}+\vec{\nabla}\times(\vec{w}\times\vec{v})=\nu\nabla^{2}\vec{w} \tag{2}$$

Now $\vec{\nabla}\times(\vec{w}\times\vec{v})=(\vec{\nabla}\cdot\vec{v})\vec{w}-(\vec{\nabla}\cdot\vec{w})\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{w}-(\vec{w}\cdot\vec{\nabla})\vec{v}$

$=(\vec{v}\cdot\vec{\nabla})\vec{w}-(\vec{w}\cdot\vec{\nabla})\vec{v}$ (Using equation of continuity $\vec{\nabla}\cdot\vec{v}=0$ and the result $\vec{\nabla}\cdot\vec{w}=\vec{\nabla}\cdot\vec{\nabla}\times\vec{v}=0$)

so that equation (2) reduces to

$$\frac{\partial \vec{w}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{w}-(\vec{w}\cdot\vec{\nabla})\vec{v}=\nu\nabla^{2}\vec{w}$$

i.e., $$\frac{\partial \vec{w}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{w}=(\vec{w}\cdot\vec{\nabla})\vec{v}+\nu\nabla^{2}\vec{w} \tag{3a}$$

i.e., $$\frac{d\vec{w}}{dt}=(\vec{w}\cdot\vec{\nabla})\vec{v}+\nu\nabla^{2}\vec{w} \tag{3b}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}$. Equation (3a) or (3b) is known as *vorticity equation*.

Dissipation of energy

We now calculate the energy which is dissipated in a viscous liquid in motion due to internal friction.

Suppose the liquid is contained within a volume V bounded by a closed surface S. The forces acting on the liquid are the external force $\vec{F}$ per unit mass, the normal pressure p on the boundary and the viscous stress acting over the surface S. Now the rate at which the work is done by these forces is

$$\int_V \rho F_i v_i dv + \int_S (T_{ij} n_j) v_i ds = \int_V \left[\rho F_i v_i + \frac{\partial}{\partial x_j}(T_{ij} v_i) \right] dv \qquad (4)$$

where T_{ij} is the stress given by

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$$

(see study Material PG(MT)05 : Group-B, page-122)

μ being the viscosity.

Let K be the kinetic energy and E be the intrinsic energy so that

$$K + E = \frac{1}{2}\int_V \rho \vec{v}.\vec{v} dv + \int_V \rho e dv,$$

e being the intrinsic energy per unit mass. Then the rate of increase of this total energy is

$$\frac{d}{dt}(K+E) = \int_V \rho\vec{v}.\frac{d\vec{v}}{dt} dv + \int_V \rho \frac{de}{dt} dv = \int_V \rho\left(v_i \frac{dv_i}{dt} + \frac{de}{dt} \right) dv \qquad (5)$$

so that from (4) and (5) we get by using the principle of energy

$$\int_V \rho\left(\frac{de}{dt} + v_i \frac{dv_i}{dt} \right) dv = \int_V \left[\rho F_i v_i + \frac{\partial}{\partial x_j}(T_{ij} v_i) \right] dv$$

$$\text{or,} \quad \int_V \left\{ \rho\frac{de}{dt} + \rho v_i \frac{dv_i}{dt} - \rho F_i v_i - \frac{\partial}{\partial x_j}(T_{ij} v_i) \right\} dv = 0.$$

Since this is true for arbitrary volume V, we must have

$$\rho \frac{de}{dt} = \rho F_i v_i - \rho v_i \frac{dv_i}{dt} + \frac{\partial}{\partial x_j}(T_{ij} v_i) \tag{6}$$

Noting that

$$\frac{\partial}{\partial x_j}(T_{ij} v_i) = T_{ij} \frac{\partial v_i}{\partial x_j} + v_i \frac{\partial T_{ij}}{\partial x_j}$$

$$= T_{ij} (e_{ij} + w_{ij}) + v_i \left(\frac{dv_i}{dt} - F_i \right) \rho$$

(see equation (4.16) in study Material PG(MT)05 : Group-B, page 66)

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{v} . \bar{\nabla}$, $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ and $w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$ and since T_{ij} are symmetric but w_{ij} are skew-symenetric so that $T_{ij}w_{ij} = 0$, we have

$$\frac{\partial}{\partial x_j}(T_{ij} v_i) = T_{ij} e_{ij} + \rho v_i \left(\frac{dv_i}{dt} - F_i \right)$$

Thus from (5), we get

$$\rho \frac{de}{dt} = \rho F_i v_i - \rho v_i \frac{dv_i}{dt} + T_{ij} e_{ij} + \rho v_i \left(\frac{dv_i}{dt} - F_i \right)$$

i.e. $$\rho \frac{de}{dt} = T_{ij} e_{ij} = \left[- p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] e_{ij}$$

$$= - pe_{ii} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) e_{ij}$$

$$= - p \frac{\partial v_i}{\partial x_i} + \frac{1}{2} \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2$$

$$\therefore \quad \rho \frac{de}{dt} = - p \frac{\partial v_i}{\partial x_i} + \Phi$$

where $\Phi = \frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2$ is the dissipation function and is necessarily positive. In Cartesian form

$$\Phi = \frac{1}{2}\mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right\}$$

Obviously, this expression is never negative and is zero only when each of the squared term vanishes. It is, therefore, evident that energy is always dissipated and reappears in the form of heat unless liquid moves without any strain, that is as a rigid body.

We now proceed to discuss the steady motion of incompressible viscous liquids through different tubes and channel.

5.1.1 Flow through tube of uniform cross-section

We consider steady flow of imcompressible viscous flow through a tube of arbitrary but uniform cross-section. We take the z-axis along the axis of the pipe. We suppose that only the non-zero velocity component is along the z-axis, so we put $u = 0$, $v = 0$, $w \neq 0$.

Under this assumption the set of basic equations are

$$\frac{\partial w}{\partial z} = 0 \quad \text{(equation of continuty)}, \tag{7}$$

$$\frac{\partial p}{\partial x} = 0 \quad \text{(equation of motion along x-direction)}, \tag{8}$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{(equation of motion along y-direction)}, \tag{9}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \text{(equation of motion along z-direction)}. \tag{10}$$

(7) implies that w is a function of x and y only and is independent of z. (8) and (9) imply that p is a function of z only. Thus (10) becomes

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz}. \tag{11}$$

L.H.S. of (11) is a function of x and y whereas R.H.S. of (11) is a function of z only. So each must be constant. We write

$$\frac{dp}{dz} = -P. \tag{12}$$

We have considered the negative sign because we expect that pressure P decreases in the direction of flow. So the equation satisfied by w is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu}. \tag{13}$$

This equation is to be solved subject to the boundary condition w = 0 on the surface of the tube. We simplify the equation by writing

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) \tag{14}$$

so that

$$\frac{\partial w}{\partial x} = \frac{\partial \psi}{\partial x} - \frac{Px}{2\mu}, \frac{\partial^2 w}{\partial x^2} = -\frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial x^2},$$

$$\frac{\partial w}{\partial y} = \frac{\partial \psi}{\partial y} - \frac{Py}{2\mu}, \frac{\partial^2 w}{\partial y^2} = -\frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial y^2}.$$

Hence,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial x^2} - \frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{P}{\mu}.$$

Since

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu}$$

we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{15}$$

Therefore, ψ satisfies the two-dimensional Laplace equation with the boundary condition

$$\psi = \frac{P}{4\mu}(x^2 + y^2) \tag{16}$$

on the surface of the tube.

5.1.2 Flow through a pipe of circular cross-section

The equation of the cross-section of the pipe is $x^2 + y^2 = a^2$, or $r = a$. Boundary condition is $\psi = \frac{P}{4\mu}a^2$ on $r = a$. To satisfy this condition we choose $\psi = A = \frac{Pa^2}{4\mu}$ = constant. Therefore the velocity w is given by,

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) = \frac{Pa^2}{4\mu} - \frac{Pr^2}{4\mu} = \frac{P}{4\pi}(a^2 - r^2).$$

Hence

$$w(r) = \frac{P}{4\mu}(a^2 - r^2). \tag{17}$$

The form (15) shows that the velocity profile is parabolic, i.e., the plot of w against r from $r = 0$ to $r = a$ is of parabolic shape. The volume rate of flow at any cross section is given by,

$$Q = \int_0^a w(r).2\pi r dr = \int_0^a \frac{P}{4\mu}(a^2 - r^2)2\pi r dr = \frac{P}{4\mu}2\pi\int_0^a (a^2 - r^2) r dr$$

$$= \frac{P\pi}{2\mu}\int_0^a (a^2 r - r^3) dr = \frac{P\pi}{2\mu}\left[a^2 . \frac{a^2}{2} - \frac{a^4}{4}\right] = \frac{P\pi a^4}{8\mu}.$$

It is clear that the pressure gradient $\frac{dp}{dz} = \frac{p_2 - p_1}{l}$, where p_1 and p_2 are the pressures at two sections at a distance l apart. So the volume rate of flow.

$$Q = \frac{\pi a^4}{8\mu}P = \frac{\pi a^4}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{\pi a^4}{8\mu}(p_1 - p_2). \tag{18}$$

This formula is used to determine the coefficient of viscosity μ. Since all other quantities can be measured experimentally, μ can be determined from the formula (16).

5.1.3 Flow through a pipe with annular cross-section

Consider the pipe $b < r < a$, i.e., the region between two concentric cylinders $r = b$ and $r = a$. The boundary conditions are $\psi = \frac{Pb^2}{4\mu}$ on the inner cylinder $r = b$ and $\psi = \frac{Pa^2}{4u}$ on the outer cylinder $r = a$.

An appropriate choice of ψ satisfying the Laplace equation in $a < r < b$ is $\psi = A + B \ln r$. By the boundary conditions, we find

$$\frac{Pb^2}{4\mu} = A + B\ln b, \quad \frac{Pa^2}{4\mu} = A + B\ln a.$$

Thus

$$B = \frac{P}{4\mu}\left(\frac{b^2 - a^2}{\ln b - \ln a}\right) = \frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)},$$

and

$$A = \frac{Pa^2}{4\mu} - B\ln a = \frac{Pa^2}{4\mu} - \left[\frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\right]\ln a.$$

Hence

$$\psi = \frac{Pa^2}{4\mu} - \left[\frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\right] + B\ln r = \frac{Pa^2}{4\mu}\left[\frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\right]\ln\left(\frac{r}{a}\right)$$

so that

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) = \psi - \frac{P}{4\mu}r^2.$$

Hence

$$w = \frac{P}{4\mu}\left[(a^2 - r^2) + (b^2 - a^2)\frac{\ln(b/a)}{\ln(r/a)}\right]. \tag{19}$$

The rate of volume flow is given by

$$Q = \int_b^a w(r).2\pi r dr = \frac{P\pi}{8\mu}\left[a^4 - b^4 - \frac{(a^2 - b^2)^2}{\ln\left(\frac{a}{b}\right)}\right]$$

5.1.4 Flow through a pipe with elliptic cross-section

Let the equation of cross section of the pipe be,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{20}$$

A suitable solution for the Laplace equation for this case is

$$\psi = A(x^2-y^2)+B. \tag{21}$$

To satisfy the boundary condition w = 0 on the surface of the pipe,

$$\psi=\frac{P}{4\mu}(x^2+y^2)=A(x^2-y^2)+B$$

on the surface of the pipe. This implies that

$$\left(\frac{P}{4\mu}-A\right)x^2+\left(\frac{P}{4\mu}+A\right)y^2=B$$

$$\text{i.e.,}\ \frac{x^2}{\dfrac{B}{\frac{P}{4\mu}-A}}+\frac{y^2}{\dfrac{B}{\frac{P}{4\mu}+A}}=1.$$

Comparing (18), (20) we obtain,

$$\frac{B}{\frac{P}{4\mu}-A}=a^2\ ;\ \frac{B}{\frac{P}{4\mu}+A}=b^2$$

$$\Rightarrow \frac{P}{4\mu}-A=\frac{B}{a^2};\ \frac{P}{4\mu}+A=\frac{B}{b^2}$$

$$\Rightarrow \frac{P}{2\mu}=B\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=B\left(\frac{a^2+b^2}{a^2b^2}\right)$$

$$\Rightarrow B=\frac{Pa^2b^2}{2\mu(a^2+b^2)}.$$

Hence,

$$A=\frac{P}{4\mu}-\frac{B}{a^2}=\frac{P}{4\mu}-\frac{Pb^2}{2\mu(a^2+b^2)}=\frac{P}{4\mu}\left[1-\frac{2b^2}{a^2+b^2}\right]=\frac{P}{4\mu}\left(\frac{a^2-b^2}{a^2+b^2}\right).$$

Hence the velocity distribution w is given by

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) = \frac{P}{4\mu}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)(x^2 - y^2) + \frac{Pa^2b^2}{2\mu(a^2 + b^2)} - \frac{P}{4\mu}(x^2 + y^2)$$

$$= \frac{Pa^2b^2}{2\mu(a^2 + b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

The rate of volume flow is give by

$$M = \iint w dxdy = \frac{Pa^2b^2}{2\mu(a^2 + b^2)}\iint\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dxdy.$$

Now we consider the ellipse x = aλ cos λ, y = bλ sin λ i.e., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$. On this ellipse, the integrand $\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 1 - \lambda^2\ (\sin^2\lambda + \cos^2\lambda) = 1 - \lambda^2$.

Now the area between this ellipse and the neighbouring ellipse (where λ is increased by λ + dλ) is

$= \pi a(\lambda + d\lambda)b(\lambda + d\lambda) - \pi a\lambda b\lambda = \pi ab(\lambda + d\lambda)^2 - \pi ab\lambda^2 = 2\pi ab\lambda d\lambda$.

Therefore

$$M = \frac{Pa^2b^2}{2\mu(a^2 + b^2)}\int_0^1(1 - \lambda^2).2\pi ab\lambda d\lambda = \frac{\pi P}{\mu}.\frac{a^3b^3}{a^2 + b^2}\int_0^1\lambda(1 - \lambda^2)d\lambda$$

$$= \frac{\pi P}{4\mu}\frac{a^3b^3}{a^2 + b^2}.$$

Now the rate of volume flow through a pipe of circular cross-section with radius $(ab)^{1/2}$ having the same cross section as the ellipse is $M_c = \frac{\pi P}{8\mu}a^2b^2$.

$$\frac{M}{M_c} = \frac{\pi P}{4\mu}\cdot\frac{a^3b^3}{a^2 + b^2} \times \frac{8\mu}{\pi P}\cdot\frac{1}{a^2b^2} = \frac{2ab}{a^2 + b^2} < 1$$

$$\Rightarrow M < M_c$$

Thus the flux through a circle is greater than that through an ellipse. The physical reason is that for a given pressure gradient the rate of flow is diminished by the friction. Now this friction is minimum on a circle because among all curves with the same enclosed area circle is the curve of minimum periphery.

5.1.5 Flow through a pipe with rectangular cross-section

Let the cross section be bounded by the planes x = a, x = − a and y = b, y = − b. We have to solve

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu} \tag{22}$$

subject to the boundary conditions,

$$\text{(i) } w = 0 \text{ at } x = a,\ x = -a,$$

$$\text{(ii) } w = 0 \text{ at } y = b,\ y = -b. \tag{23}$$

One particular solution of (22) satisfying the boundary condition is

$$w_1 = \frac{P}{2\mu}(a^2 - x^2). \tag{24}$$

If we write, $w = w_1 + w_2$ then,

$$\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} = 0. \tag{25}$$

We solve this equation (25) by method of separation of variables where we assume

$$w_2(x, y) = X(x)Y(y). \tag{26}$$

Substituting (26) in (25) we get,

$$\frac{d^2X(x)}{dx^2}Y(y) + X(x)\frac{d^2Y(y)}{dy^2} = 0$$

$$\Rightarrow -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = C_n^2$$

$$\Rightarrow \frac{d^2X(x)}{dx^2} = -C_n^2X(x);\ \frac{d^2Y(y)}{dy^2} = -C_n^2Y(y).$$

Solutions are

$$X(x) = A\cos(C_nx) + B\sin(C_nx),\quad Y(y) = A\cos(C_ny) + B\sin(C_ny).$$

Now, from the symmetry of the cross section with respect to both x and y, it follows that w must be even function of x and y. Since, from (3) w_1 is already even in x, w_2 must be even in x, w_2 must be even in x and y. Hence

$$B = D = 0,$$

and therefore

$$w_2 = \sum_{n=0}^{\infty} A_n \cos(C_n x)\cosh(C_n y).$$

Here,

$$w = w_1 + w_2 = \frac{P}{4\mu}(a^2 - x^2) + \sum_{n=0}^{\infty} A_n \cos(C_n x)\cosh(C_n y).$$

To satisfy the boundary condition w = 0 at x = a, x = − a, we have

$$0 + \sum_{n=0}^{\infty} A_n \cos(C_n a)\cosh(C_n y)$$

$$\Rightarrow \cos(C_n a) = 0, \text{ i.e., } C_n a = (2n+1)\frac{\pi}{2}, \text{ i.e., } C_n = (2n+1)\frac{\pi}{2a}$$

Therefore

$$w = \frac{P}{2\mu}(a^2 - x^2) + \sum_{n=0}^{\infty} A_n \cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi y}{2a}\right\},$$

By the boundary condition (ii) w = 0 at y = b, y = − b and we have,

$$0 = \frac{P}{2\mu}(a^2 - x^2) + \sum_{n=0}^{\infty} A_n \cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}$$

$$\Rightarrow -\frac{P}{2\mu}(a^2 - x^2) = \sum_{n=0}^{\infty} A_n \cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}.$$

Multiplying both sides by $\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}$ and integrating between −a and a, we get,

$$-\frac{P}{2\mu}\int_{-a}^{a}(a^2-x^2)\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}dx$$

$$=A_n\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}\int_{-a}^{a}\cos^2\left\{(2n+1)\frac{\pi x}{2a}\right\}dx$$

$$\Rightarrow -\frac{P}{2\mu}\left[\frac{2a^3}{\pi(2n+1)}\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}-\frac{2a}{\pi(2n+1)}x^2\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}\right]_{-a}^{a}$$

$$-\frac{P}{2\mu}\left[\frac{4a}{\pi(2n+1)}\int_{-a}^{a}x\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}dx\right]$$

$$=\frac{A_n}{2}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}\int_{-a}^{a}\left\{1+\cos^2(2n+1)\frac{\pi x}{2a}\right\}dx$$

$$\Rightarrow -\frac{P}{2\mu}\left[-\frac{8a^2}{\pi^2(2n+1)}x\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}+\frac{16a^3}{\pi^3(2n+1)^3}\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}\right]_{-a}^{a}$$

$$=\frac{A_n}{2}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}.2a$$

$$\Rightarrow -\frac{P}{2\mu}\frac{32a^3}{\pi^3(2n+1)^3}(-1)^n=aA_n\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}$$

$$\Rightarrow A_n=-\frac{P}{2\mu}\times\frac{32a^2(-1)^n}{\pi^3(2n+1)^3}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}.$$

Hence

$$w=\frac{P}{2\mu}(a^2-x^2)-\frac{P}{\mu}\sum_{n=0}^{\infty}\frac{16a^2(-1)^n}{\pi^3(2n+1)^3}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}$$

$$\times\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi y}{2a}\right\}.$$

5.2 Boundary Layer

5.2.1 Concept of boundary layer

The concept of boundary layer was introduced by Prandtl in 1904. He assumed that for fluid with small viscosity, the flow around a solid body can be divided into two parts

(i) a very thin layer called boundary, adjacent to the boundary layer where viscous effect is important and

(ii) a region outside the boundary where viscous effect is not important the flow may be taken as potential flow. Within this boundary layer, the Navier-Stokes equation can be simplified. These are called the boundary layer equations.

5.2.2 Two dimensional boundary layer flow over a plane wall

For motion in the (x, y)-plane, the Navier-Stokes equation and the equation of continuity are,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{27}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{28}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{29}$$

Here x-axis is taken along the wall and y-axis normal to the wall. Due to no slip condition, $u = v = 0$ at the wall. Let $U(x, t)$ be the velocity outside the boundary layer. Then the velocity component u within the boundary layer rises rapidly from its value 0 at the wall to the value U at a small distance $\delta(x)$. δ is the boundary layer thickness and $\delta \ll 1$. We now calculate the order of magnitude of viscous terms in the equation of motion. We take u, x, t are of $O(1)$, but $y = O(\delta)$. By the equation of continuity,

$$\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}=O(1)$$

so that

$$v = O(\delta).$$

Then from (1),

$$\frac{\partial u}{\partial t} = O(1),\ \frac{\partial u}{\partial x} = O(1),\ v\frac{\partial v}{\partial y} = O(\delta).O\left(\frac{1}{\delta}\right) = O(1),$$

$$\frac{\partial^2 u}{\partial x^2} = O(1),\ \frac{\partial^2 u}{\partial y^2} = O\left(\frac{1}{\delta^2}\right).$$

Thus in equation (27) we can neglect the term $\frac{\partial^2 u}{\partial x^2}$ compared to $\frac{\partial^2 u}{\partial y^2}$. So equation (27) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{30}$$

If each term of this equation is of the same order of magnitude, we note that,

$$\nu = O(\delta^2),\ \Rightarrow \delta = O(\sqrt{\nu}).$$

Now we consider equation (28). We see that,

$$\frac{\partial v}{\partial t} = O(\delta),\ u\frac{\partial v}{\partial x} = O(\delta),\ v\frac{\partial v}{\partial y} = O(\delta),\ \frac{\partial^2 v}{\partial x^2} = O(\delta),\ \frac{\partial^2 v}{\partial y^2} = O\left(\frac{1}{\delta}\right)$$

Therefore,

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = O(\delta). \tag{31}$$

Thus the pressure gradient normal to the wall is of the order of δ. Hence integrating (31) with respect to y from $y = 0$ to δ, the pressure p may be neglected. Thus within the boundary layer pressure p may be taken as a function of x only and is given by its value at the outer edge of the boundary layer. Suppose that the flow outside the boundary layer is given by U(x, t). Then,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}.$$

Thus Prandtl's boundary layer equations are,

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2}, \tag{32a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{32b}$$

Boundary conditions are u = v = 0 at y = 0 and u = U(x, t), at $y \to \infty$. The first boundary condition is the usual no slip condition.

It may be seen that considerable simplification has been achieved in the above equations which consist of two equations with two unknowns u and v. However the equations are still nonlinear, therefore it has been possible to solve the equations directly only for a limited number of problems, such as flow past a flat plate.

5.2.3 Boundary layer over a flat plate : (Blasius Solution)

The first application of Prandtl's boundary layer equations was made by H. Blasius (1908) to determine analytically an expression for thickness of the boundary layer over a wide semi infinite plate.

Now consider the steady flow of viscous incompressible fluid past a semi infinite plate placed in the direction of the uniform stream with velocity U.

We take the origin at the leading edge of the plate, x-axis along the plate and y-axis normal to the plate. In this case, the potential flow outside the boundary layer equations are,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 U}{\partial y^2} \quad \left[\text{since}, \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \text{ as } U = \text{constant}\right], \tag{33a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{33b}$$

with the boundary condition u = v = 0 at y = 0 and u = U at $y \to \infty$.

The equation of continuity can be integrated introducing by the stream function $\psi(x, y)$,

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}. \tag{34}$$

The characteristic parameters of this flow are U, x, y and v. i.e., the problem is determined in terms of these parameters. We may write

$u = U\,F(\eta)$ where $\eta = \left(\frac{U}{vx}\right)^{1/2} y$ $\left[\text{Here } \delta = \sqrt{\frac{vx}{U}} \text{ is the boundary layer thickness}\right]$.

Now by the first relation of (33), we get,

$$\psi = \int u dy = U \int F(\eta) \left(\frac{\nu x}{U} \right)^{1/2} d\eta = (\nu x U)^{1/2} \int F(\eta) d\eta$$

$$= (\nu x U)^{1/2} f(\eta) \qquad \left[\text{where, } f(\eta) = \int F(\eta) d\eta \right].$$

Now,

$$u = \frac{\partial \psi}{\partial y} = (\nu x U)^{1/2} \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} = (\nu x U)^{1/2} \left(\frac{U}{\nu x} \right)^{1/2} f'(\eta) = U f'(\eta)$$

and

$$v = -\frac{\partial \psi}{\partial x} = -(\nu U)^{1/2} \frac{1}{2} x^{-1/2} f(\eta) - (\nu x U)^{1/2} \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$= -\frac{1}{2} \left(\frac{\nu U}{x} \right)^{1/2} f(\eta) - (\nu x U)^{1/2} f'(\eta) \left(-\frac{1}{2} \right) \left(\frac{U}{\nu} \right)^{1/2} \frac{y}{x^{3/2}}$$

$$= -\frac{1}{2} \left(\frac{\nu U}{x} \right)^{1/2} f(\eta) + \frac{1}{2} (\nu x U)^{1/2} f'(\eta) . \frac{\eta}{x}.$$

Hence,

$$v = \frac{1}{2} \left(\frac{\nu U}{x} \right)^{1/2} \left\{ -f(\eta) + \eta f'(\eta) \right\},$$

$$\frac{\partial u}{\partial x} = U f'(\eta) \frac{\partial \eta}{\partial x} = -\frac{1}{2} U \frac{\eta}{x} f''(\eta),$$

$$\frac{\partial u}{\partial y} = U f''(\eta) \frac{\partial \eta}{\partial y} = U f''(\eta) \left(\frac{U}{\nu x} \right)^{1/2} = U \left(\frac{U}{\nu x} \right)^{1/2} f''(\eta),$$

$$\frac{\partial^2 u}{\partial y^2} = U \left(\frac{U}{\nu x} \right)^{1/2} f'''(\eta) \frac{\partial \eta}{\partial y} = U \left(\frac{U}{\nu x} \right)^{1/2} f'''(\eta) \left(\frac{U}{\nu x} \right)^{1/2} = \frac{U^2}{\nu x} f'''(\eta).$$

Substituting all these terms into equation (1) we get,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow Uf'(\eta)\left(-\frac{1}{2}\right)U\frac{\eta}{x}f''(\eta)+\frac{1}{2}\left(\frac{vU}{x}\right)^{1/2}$$

$$\{-f(\eta)+\eta f'(\eta)\}U\left(\frac{U}{vx}\right)^{1/2}f''(\eta)=v\frac{U^2}{vx}f'''(\eta)$$

$$\Rightarrow -\frac{1}{2}\frac{U^2}{x}\eta f'f''+\frac{1}{2}\frac{U^2}{x}f''(\eta f'-f)=\frac{U^2}{x}f'''$$

$$\Rightarrow 2f'''=-\eta f'f''+f'(\eta f'-f)\Rightarrow 2f'''+ff''=0$$

$$\Rightarrow 2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=0. \qquad (35)$$

So the boundary conditions are

(i) $f = f' = 0$ at $\eta = 0$ (i.e., $y = 0$),

(ii) $f' = 1$ as $y \to \infty$.

Since the equation (35) is a nonlinear equation, its solution in closed form is not possible. Blasius solved it by power series expansion of $f(\eta)$ about $\eta = 0$. We assume,

$$f(\eta)=A_0+A_1\eta+\frac{A_2}{2!}\eta^2+\frac{A_3}{3!}\eta^3+\cdots$$

By the boundary conditions

(i) $f = 0$ at $\eta = 0 \Rightarrow A_0 = 0$

(ii) $f' = 0$ at $\eta = 0 \Rightarrow A_1 = 0$.

Substituting these in the differential equation (35), we get,

$$f(\eta)=\frac{A_2}{2!}\eta^2+\frac{A_3}{3!}\eta^3+\cdots$$

$$f'(\eta)=A_2\eta+\frac{A_3}{2}\eta^2+\frac{A_4}{6}\eta^3+\frac{A_5}{24}\eta^4+\frac{A_6}{120}\eta^5\cdots$$

$$f''(\eta)=A_2+A_3\eta+\frac{A_4}{2}\eta^2+\frac{A_5}{6}\eta^3+\frac{A_6}{24}\eta^4+\cdots$$

$$f'''(\eta)=A_3+A_4\eta+\frac{A_5}{2}\eta^2+\frac{A_6}{6}\eta^3+\cdots$$

Therefore,

$$2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=0$$

$$\Rightarrow 2\left(A_3+A_4\eta+\frac{A_5}{2}\eta^2+\frac{A_6}{6}\eta^3+\cdots\right)+\left(\frac{A_2}{2!}\eta^2+\frac{A_3}{3!}\eta^3+\cdots\right)$$

$$\times\left(A_2+A_3\eta+\frac{A_4}{2}\eta^2+\frac{A_5}{6}\eta^3+\frac{A_6}{24}\eta^4+\cdots\right)=0$$

$$\Rightarrow\left(2A_3+2A_4\eta+A_5\eta^2+\frac{A_6}{3}\eta^3+\cdots\right)+\left(\frac{A_2^2}{2}\right)\eta^2+\left(\frac{A_2A_3}{4}+\frac{A_2A_3}{2}\right)\eta^3+\cdots=0$$

$$\Rightarrow 2A_3+2A_4\eta+\left(A_2^2+2A_5\right)\frac{\eta^2}{2}+\left(\frac{A_6}{3}+\frac{3A_2A_3}{4}\right)\eta^3+\cdots=0.$$

Since, coefficients of various powers of η vanish separately,

$$A_3=0,\ A_4=0,\ A_5=-\frac{1}{2}A_2^2,\ A_6=0,\cdots$$

$$f(\eta)=\frac{A_2}{2!}\eta^2-\frac{1}{2}\frac{A_2^2}{5!}\eta^5+\frac{1}{4}.\frac{11}{81}.A_2^3\eta^8+\cdots$$

$$=A_2^{1/3}\left\{\frac{1}{2!}\left(A_2^{1/3}\eta\right)^2-\frac{1}{2}\frac{1}{5!}\left(A_2^{1/3}\eta\right)^5+\frac{1}{4}.\frac{11}{81}.\left(A_2^{1/3}\eta\right)^8+\cdots\right\}=A_2^{1/3}F\left(A_2^{1/3}\eta\right)$$

where,

$$F\left(A_2^{1/3}\eta\right)=\frac{1}{2!}\left(A_2^{1/3}\eta\right)^2-\frac{1}{2}.\frac{1}{5!}\left(A_2^{1/3}\eta\right)^5+\cdots$$

Now by the boundary condition that $f'\to 1$ at $\eta\to\infty$, we get,

$$1=\lim_{\eta\to\infty}\left[A_2^{2/3}F'\left(A_2^{1/3}\eta\right)\right]$$

$$\text{i.e., } A_2=\left[\frac{1}{\lim\limits_{\eta\to\infty}F'(\eta)}\right]^{3/2}$$

The value of A_2 can be obtained numerically. Howarth found that $A_2 = .332$. This completes the solution which is also known as Blasius solution.

5.2.4 Shearing stress on the plate

The shearing stress on the surface of the plate can be calculated with the help of the above solution. The shearing stress is given by

$$\tau_{\omega} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U\left(\frac{U}{\nu x}\right)^{1/2} f''(0) = 0.332\,\mu U\left(\frac{U}{\nu x}\right)^{1/2}$$

5.2.5 Boundary layer thickness

Although the velocity u reaches the potential value U asymptotically, a value which is very near to U is attained within a small distance δ. A measure of this boundary layer thickness is introduced by the following relation

$$U\delta = \int_0^{\infty}(U-u)\,dy\,.$$

The right hand side signifies the decrease in the flow rate due to friction within the boundary layer and the L.H.S. represents the total potential flow that has been displaced from the wall. So δ represents the distance to which the free stream has been displaced due to boundary layer. This δ is called displaced thickness. From a flat plate this given by

$$\delta = \int_0^{\infty}\left(1-\frac{u}{U}\right)dy\,.$$

The upper limit of integration is taken as $y = \infty$, because the integrand becomes zero outside the boundary layer.

5.3 Model Questions

Short Questions :

1. What is the difference between an ideal (non-viscous) and a real (viscous) fluid?
2. What is meant by dissipation?
3. Discuss the concept of boundary layer.
4. Define boundary layer thickness. What is its significance?

Broad Questions :

1. Deduce the vorticity equation for an incompressible viscous fluid.
2. Show that a viscous liquid cannot move without dissipation of energy by viscosity unless it moves as if rigid.
3. Discuss the motion of an incompressible viscous fluid through (i) a tube of circular, annular, elliptic and rectangular cross-section.
4. Deduce the equations of motion for two-dimensional boundary layer over a plane wall.
5. Find the Blasius solution for the two-dimensional boundary layer flow over a flat plate.

5.4 Summary

In this chapter, some properties of an incompressible viscous fluid are introduced and the motion of this fluid through tubes of different cross-section has been discussed. The concept of boundary layer and its property are also outlined.

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